Shree H.N. Shukla College of Science
M.Sc. (Mathematics) Sem-4
Prelims Test
CMT-4003: Number Theory-2

[Time: 2.5 Hours] [Total Marks: 70]

Q-1 Answer any seven **[$7 \times 2 = 14$]**

1. Define: (a) Primitive Pythagorean triplet (b) Simple Continued fraction expansion.
2. Write down all Farey Fractions between 0 and 1 up to 7^{th} row.
3. Express $\frac{2018}{17}$ as a simple continued fraction.
4. Find the continued expansion of $\frac{\sqrt{5}+1}{2}$ and $\frac{\sqrt{5}-1}{2}$.
5. Find atleast three positive solutions of the equation $x^2 - 2y^2 = 1$.
6. Find two Primitive Pythagorean triplet (x, y, z) for which $z < 59$.
7. Find the value of $\langle 1, 1, 2, 2, 2, \dots \dots \rangle$ and $\langle -2, 2, 4, 3, 3, \dots \dots \rangle$.
8. If r and s are positive integers and t is a rational solution of $x^r = s$ then t must be_____. Justify your answer.
9. Show that $\gcd(x, y) = \gcd(y, z)$, where (x, y, z) is a Pythagorean Triplet.
10. Find two positive integers n such that $1 + 2 + 3 + 4 + \cdots + n$ is a perfect square.

Q-2 Answer any two of the following. **[$2 \times 7 = 14$]**

1. Suppose θ is an irrational and $\frac{a}{b}$ is a rational number such that $\left|\theta - \frac{a}{b}\right| < \frac{1}{2b^2}$ then prove that $\frac{a}{b} = r_n$ for some n.
2. State and prove the necessary and sufficient condition under which the continued Farey expansion of a quadratic irrational is purely periodic.
3. Write an algorithm to find the sequence of integers $a_0 a_1 a_2, \dots a_n \dots$ when an irrational number is given and then explain with an example.

Q-3 Answer the following. **[$2 \times 7 = 14$]**

1. Prove that for each $n > 0$ there is a polynomial $f_n(x)$ of degree n, leading coefficient 1 and with integer coefficients such that $f_n(2\cos\theta) = \mathrm{a}\cos n\theta$.
2. Suppose θ is an irrational number whose continued fraction expansion is periodic. Prove that θ is quadratic irrational.

OR

Q-3 Answer the following. **[2 × 7 = 14]**

1. Prove that π is irrational using elementary method.
2. If $\frac{h_j}{k_j}$ denotes j[th] convergent of an irrational number θ then prove that for all $n \geq 1$

   $a) \mid \theta k_n - h_n \mid < \mid \theta k_{n-1} - h_{n-1} \mid.$

   $b) \left| \theta - \frac{h_n}{k_n} \right| < \left| \theta - \frac{h_{n-1}}{k_{n-1}} \right|.$

Q-4 Answer the following. **[2 × 7 = 14]**

1. If θ is a quadratic irrational such that (i) $\theta > 1$ (ii) $-1 < \theta' < 0$ then prove that continued fraction expansion of θ is purely periodic.
2. Prove that there are infinitely many positive integers n such that $1 + 2 + 3 + 4 + \cdots + n = m^2$ for some integer m.

Q-5 Answer any two of the following. **[2 × 7 = 14]**

1. a) Show that the equation $x^2 = y^2 + 7$ has no solution in integer.

   b) Find four solution of the equation $x^2 - 29y^2 = 1$.
2. Find general solutions (if any) of the following Diophantine equation:

   (1) $2\mathrm{x} + 9\mathrm{y} = 11$

   (2) $100\mathrm{x} + 101\mathrm{y} = 2018$.
3. Show that if the triplet $(\mathrm{x}, \mathrm{y}, \mathrm{z})$ is a Primitive Pythagorean triplet then there exists r and s such that $\mathrm{r} > \mathrm{s} \geq 1$. $(\mathrm{r}, \mathrm{s}) = 1$ and r is even then s is odd and vice-versa.