SHREE H. N. SHUKLA GROUP OF COLLEGES

S. Y. B. Sc. SEM – IV

Subject: Mathematics

Paper – 401

Unit – 2

Title Explanation

INFINITE SERIES

- The **sum** of infinite terms that follow a rule.

When we have an infinite **sequence** of values:

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots \ldots \ldots \ldots \ldots$$

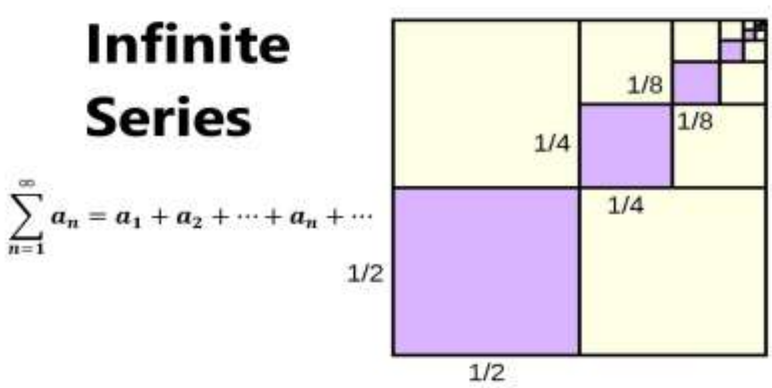
Which follow a rule (in this case each term is half the previous one), and we **add them all up**:

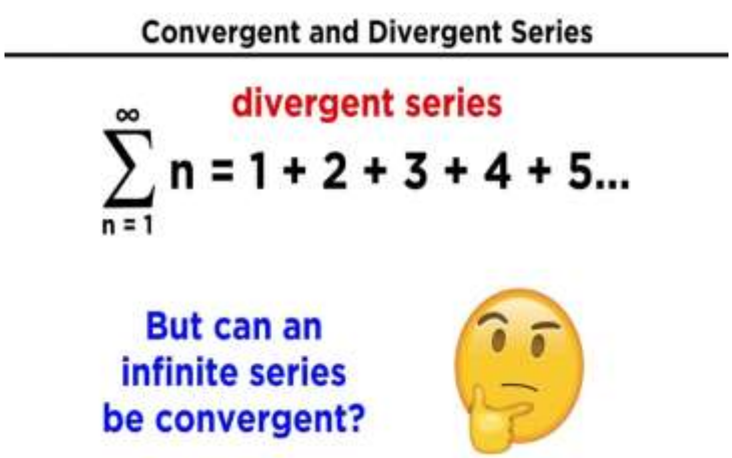
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots \ldots \ldots \ldots \ldots . = S$$

We get an **infinite series**.

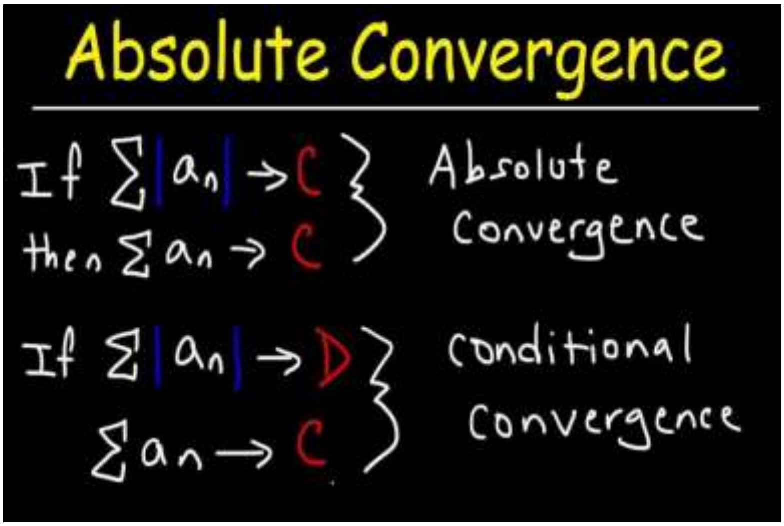
- A "**SERIES"** sound like it is the **list of numbers**, but it is actually when we add them together.

Trailer of Topic






Sum of an infinite
geometric series


Absolute Convergence
If Σ|aₙ| → C
then Σ aₙ → C
Absolute Convergence
If Σ|aₙ| → D
Σ aₙ → C
Conditional Convergence


Alternating Series
Test of Convergence


Power Series
Power series
radius & interval
of convergence

Learning Outcome

- ☑ Infinite series are useful in mathematics and in such disciplines as physics, chemistry, biology, and engineering.

- ☑ Explain the meaning of the sum of an infinite series.

- ☑ Calculate the sum of a geometric series.

- ☑ Evaluate a telescoping series.

❖ Definition: Infinite Series

- If $\{u_{1,}u_2u_{3,}\ldots\ldots\ldots\ldots..u_{n,}\ldots\ldots\ldots\ldots\ldots..\}$ is any sequence then the **infinite sum** $u_{1,}+u_2+u_{3,}+\cdots\ldots\ldots..+u_{n,}+\cdots\ldots\ldots\ldots\ldots..$ is called infinite series OR simply series.
- This is denoted by $\sum_{n=1}^{\infty}\boldsymbol{u_n}$ OR $\sum \boldsymbol{u_n}$
- If the **number of terms is finite** then the series is called finite series and if the **number of terms is unlimited** then it is called an infinite series.

❖ Examples:

(1) $\sum n^2 = 1^2 + 2^2 + 3^2 + \ldots\ldots\ldots\ldots\ldots. + n^2 + \ldots\ldots\ldots\ldots\ldots..$

(2) $\sum(-1)^n = 1 - 1 + 1 - 1 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots..$

(3) $\sum\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots\ldots\ldots\ldots + \frac{1}{n} + \ldots\ldots\ldots\ldots....$

(4) $\sum(2n-1) = 1 + 3 + 5 + \ldots\ldots\ldots\ldots\ldots.. + (2n+1) + \ldots\ldots\ldots..$

- A series $\sum u_n$ is said to be a series of positive terms **If** $\boldsymbol{u_n > 0, \forall n \in N}$**.**
- A series is said to be an alternating series if the terms of the series are **alternately positive OR negative.**
- The series (1), (3) and (4) in the above examples are series of positive terms, whereas series (2) is alternating series.

Sr. No.	Question	Answer
1	Infinite sum of any sequence is called ………	**Infinite series**
2	Give the condition of series of positive terms.	**$u_n > 0, \forall n \in N$**
3	Which type of terms held in Alternating series?	**alternately positive OR negative**

❖ Definition: Sequence of partial sums of a series

- Let $\sum u_n$ be any series.
- Let us consider the following sums:

$$s_1 = u_1$$

$$s_2 = u_1 + u_2$$

$$s_3 = u_1 + u_2 + u_3$$

…………………

…………………

$$\boldsymbol{s_n = u_1 + u_2 + u_3 + \ldots\ldots\ldots\ldots\ldots.. + u_n}$$

- Let $\sum u_n$ be a given series and $\mathbf{s_1, s_2, s_3, \ldots\ldots\ldots\ldots.., s_n, \ldots\ldots\ldots\ldots\ldots\ldots}$ be a **different partial sums of** $\boldsymbol{\sum u_n}$. The sequence $\boldsymbol{\{s_1, s_2, s_3, \ldots\ldots\ldots\ldots.., s_n, \ldots\ldots\ldots\ldots\ldots\ldots\}}$ is called the sequence of partial sums of $\sum u_n$.

Sr. No.	Question	Answer
1	The sequence $\{s_1, s_2, s_3, \ldots\ldots\ldots\ldots.., s_n, \ldots\ldots\ldots\ldots\ldots\ldots\}$ is called	**the sequence of partial sums of** $\sum u_n$
2	Give the value of s_2.	$s_2 = u_1 + u_2$

❖ Definition: Convergent Series

- Let $\sum u_n$ be a series and $\{s_n\}$ be the corresponding sequence of partial sums.
- The series $\sum u_n$ is convergent, if the sequence $\{s_n\}$ is convergent.

i.e. $\lim_{n\to\infty} s_n = l = \textit{finite and unique}.$

- In this case we say $\sum u_n$ **converges to l.**

❖ Definition: Divergent Series

- Let $\sum u_n$ be a series and $\{s_n\}$ be the corresponding sequence of partial sums.
- The series $\sum u_n$ is divergent, if the sequence $\{s_n\}$ is divergent.

i.e. $\lim_{n\to\infty} s_n = +\infty \; OR \; -\infty$

❖ Definition: Oscillatory Series

- Let $\sum u_n$ be a series and $\{s_n\}$ be the corresponding sequence of partial sums.

  (a) The series $\sum u_n$ is said to be **oscillate finitely**, if the sequence $\{s_n\}$ is Oscillates finitely.

I.e. $\lim_{n\to\infty} s_n$ **is finite but not unique.**

  (b) The series $\sum u_n$ is said to be **oscillate infinitely**, if the sequence $\{s_n\}$ is oscillates infinitely.

  i.e. $\lim_{n\to\infty} s_n = +\infty \; OR -\infty$

Sr. No.	Question	Answer
1	Limit of a sequence is finite and unique then it is said to be………	**Convergent series**
2	Write down the condition for divergent series.	$\lim\limits_{n\to\infty} s_n = +\infty \; OR -\infty$
3	Limit of a sequence is finite but not unique then it is said to be………	**Oscillate finite series**
4	How many types have Oscillatory series?	**2 types**
5	Give the name of two types of Oscillatory series.	**Oscillates finitely & Oscillates infinitely**

- **NOTE:**

  Series which divergent OR oscillates are often classed as non convergent series.

EXAMPLE-1:

Show that the series $\frac{1}{1*2}+\frac{1}{2*3}+\frac{1}{3*4}+\ \dots\dots\dots..+\frac{1}{n(n+1)}+\ \dots\dots\dots\dots$ is convergent.

SOLUTION:

Since $u_n=\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$

$$u_1=\frac{1}{1}-\frac{1}{2}$$

$$u_2=\frac{1}{2}-\frac{1}{3}$$

$$u_3=\frac{1}{3}-\frac{1}{4}$$

....

....

$$\mathbf{u_n=\frac{1}{n}-\frac{1}{n(n+1)}}$$

$\therefore S_n=u_1+u_2+u_3+\ \dots\dots\dots..+u_n=1-\frac{1}{n+1}$

$\therefore \boldsymbol{\lim_{n\to\infty} S_n}=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=\mathbf{1}$, which is **unique and finite.**

Hence, $\sum u_n$ is **convergent and converges to 1.**

EXAMPLE-2:

Show that the series $\sum(-1)^n$ oscillates finitely.

SOLUTION:

Here $u_n = (-1)^n$

$\therefore S_n = 1 - 1 + 1 - 1 + \ldots\ldots\ldots\ldots\ldots up\ to\ n\ terms$

= 1 OR 0 according to as n is odd OR even.

$\therefore \mathbf{lim}_{n\to\infty} \boldsymbol{S_n} = \mathbf{1}\ \boldsymbol{OR}\ \mathbf{0}$, which is **finite but not unique.**

Hence**,** $\sum(-\mathbf{1})^{\boldsymbol{n}}$ **oscillates finitely**.

EXAMPLE-3:

Show that the series 1+2+3+ ………. +n+ ………. Diverges.

SOLUTION:

Since $S_n = 1 + 2 + 3 + \ldots\ldots\ldots\ldots.. + n = \frac{n(n+1)}{2}$

$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n(n+1)}{2} = \infty$

$\therefore \{\boldsymbol{S_n}\}$ **is divergent.**

Hence, $\sum u_n$ is **divergent.**

- **NOTE:**
  - In the above examples we have determined the **nature of the series directly using the definition.**
  - In this universe of series, we come across with it is either difficult OR impossible to compute the sequence of partial sums.
  - Later we come across few sufficient conditions, known as the **tests for convergence, which will be useful to determine the nature of the given series.**

Theorem-1:

A series of positive terms either **converges OR diverges to** $+\infty$**.**

Theorem-2:

If a series $\sum u_n$ **is convergent** then $\lim_{n\to\infty} u_n = 0$**.**

- **NOTE:**
  - The **convergence of the above theorem is not true** in general.

I.e. if $\lim_{n\to\infty} u_n = 0$

Then the corresponding series $\sum u_n$ is not convergent in general.

Consider the series $\sum \frac{1}{\sqrt{n}}$

Here, $u_n = \frac{1}{\sqrt{n}}$

$\therefore \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

We consider,

$$S_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ \ldots.. + \frac{1}{\sqrt{n}}$$
$$> \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \ \ldots\ldots + \frac{1}{\sqrt{n}}(n - times)$$
$$= n * \frac{1}{\sqrt{n}} = \sqrt{n}$$

$\therefore \lim_{n\to\infty} S_n > \lim_{n\to\infty} \sqrt{n} = \infty$

Thus $\sum \boldsymbol{u_n}$ **is divergent even though** $\boldsymbol{\lim_{n\to\infty} u_n = 0}$

- **NOTE:**

  (i) $\sum u_n$ convergent $\Rightarrow \lim_{n\to\infty} u_n = 0$

  (ii) $\lim_{n\to\infty} u_n = 0 \Rightarrow \sum u_n$ may be convergent OR not.

  (iii) $\lim_{n\to\infty} u_n \neq 0 \Rightarrow \sum u_n$ is not convergent.

  (iv) If $\sum u_n$ is a series of positive terms and $\lim_{n\to\infty} u_n \neq 0$ then $\sum u_n$ diverges to ∞.

Sr. No.	Question	Answer
1	A series of positive terms either converges OR diverges to………	$+\infty$
2	If a series $\sum u_n$ is convergent then……..	$\boldsymbol{\lim_{n\to\infty} u_n = 0}$
3	If $\lim_{n\to\infty} u_n = 0$ then the series is convergent.(T/F)	**False**
4	If $\sum u_n$ is a series of positive terms and $\lim_{n\to\infty} u_n \neq 0$ then $\sum u_n$ diverges to ∞.(T/F)	**True**

❖ Cauchy's General Principal of Convergence:

A necessary and sufficient condition for the series $\sum u_n$ to convergent is that, given ε>0, however small, there exist m∊N, such that

$$\left|\boldsymbol{s_{n+p}} - \boldsymbol{s_n}\right| < \varepsilon, \forall n \geq m, p \in N$$

OR

$$\left|\boldsymbol{u_{n+1}} + \boldsymbol{u_{n+2}} + \ \dots \dots \dots \dots + \boldsymbol{u_{n+p}}\right| < \varepsilon, \forall n \geq m, p \in N$$

❖ Definition: Geometric series

The series $\sum_{n=0}^{\infty} \boldsymbol{r^n} = \mathbf{1} + \boldsymbol{r} + \boldsymbol{r^2} + \ \dots \dots \dots \dots + \boldsymbol{r^{n-1}} + \ \dots \dots \dots \dots$ is called the Geometric series.

Theorem-3:

The Geometric series $\sum_{n=1}^{\infty} r^n$ is

(i) Convergent if **|r|<1,**

I.e. -1<r<1 and its sum is $\frac{1}{1-r}$

(ii) Divergent if **r≥1**

(iii) Finitely oscillating if **r=-1**

(iv) Infinitely oscillating if **r<-1**

EXAMPLE-4:

Test the convergence of $\sum_{n=0}^{\infty} \frac{3^{2n}}{2^{3n}}$

SOLUTION:

Here, $u_n = \frac{3^{2n}}{2^{3n}} = \left(\frac{3^2}{2^3}\right)^n = \left(\frac{9}{8}\right)^n$

$\therefore \sum u_n = 1 + \frac{9}{8} + \left(\frac{9}{8}\right)^2 + \ldots\ldots\ldots\ldots\ldots$

Which is Geometric series with $\boldsymbol{r} = \frac{\mathbf{9}}{\mathbf{8}} > 1$

Hence, $\sum \boldsymbol{u_n}$ **is divergent.**

Sr. No.	Question	Answer
1	Write down the Cauchy's general principle of convergence in mathematical form.	$\lvert s_{n+p} - s_n \rvert < \varepsilon,$ $\forall n \geq m, p \in N$
2	The series $\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \ldots\ldots\ldots\ldots + r^{n-1} + \ldots\ldots\ldots\ldots$ is called	**Geometric series**
3	Give the general form of Geometric series.	$\sum_{n=0}^{\infty} r^n$
4	In Geometric series, If we get r ≥ 1 then the given series is…….	**Divergent**
5	Give the example of Geometric series.	$1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} - \ldots$

❖ Definition: Alternating series

A series of the type $\boldsymbol{u_1 - u_2 + u_3 - u_4 + \ldots\ldots\ldots\ldots\ldots\ldots;}$
Where $u_n > 0, \forall n \in N$, is called an alternating series and it is denoted by $\sum_{n=1}^{\infty}(-1)^{n+1} u_n$.For example, the series **3-9+27-28+ ……………**and $\frac{1}{4} - \frac{5}{7} + \frac{7}{10} - \frac{9}{13} + \ldots\ldots\ldots\ldots$ are alternating series.

❖ Libnitz test for the convergence of an Alternating series:

The infinite series $u_1 - u_2 + u_3 - u_4 + \dots\dots\dots\dots\dots$ in which terms are alternatively positive and negative is convergent if **each term is numerically less than its preceeding term** and $\lim_{n\to\infty} u_n = 0$.

EXAMPLE-5:

Show that Alternating series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots\dots\dots\dots\dots$ is convergent.

SOLUTION:

By comparing $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots\dots\dots\dots\dots = a_1 - a_2 + a_3 - a_4 + a_5 - \dots\dots$

We see that $a_1 = 1, a_2 = \frac{1}{3}, a_3 = \frac{1}{5}, a_4 = \frac{1}{7}, a_5 = \frac{1}{9}, \dots\dots\dots\dots\dots$

$\therefore \boldsymbol{a_1 > a_2 > a_3 > a_4 > a_5 > \dots\dots\dots\dots}$

$\therefore$ Each term is numerically less than its preceeding term.

Now,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2n-1} = 0$$

Hence by Leibnitz's test the given series is **convergent.**

Sr. No.	Question	Answer
1	A series of the type $u_1 - u_2 + u_3 - u_4 + \ldots\ldots;$ Where $u_n > 0, \forall n \in N$, is called……..	**Alternating series**
2	Give an example of Alternating series.	$\frac{1}{4} - \frac{5}{7} + \frac{7}{10} - \frac{9}{13} + \ldots$
3	Which test used for the convergence of Alternating series?	**Leibnitz's test**

❖ Test for convergence:

We have proved that a series converges by actually finding its sum. However, for most convergent series the exact sum s_n is difficult OR impossible to find. In this case we consider the following standard tests;

(i) Comparision test
(ii) P-test
(iii) D'-Alemert's ratio test
(iv) Raabe's test

(i) Comparision test:

Test-1:

Let $\sum u_n$ and $\sum v_n$ be any two series of positive terms.

(A) If $\boldsymbol{u_n \leq v_n}$, for all n≥1 and $\sum \boldsymbol{v_n}$ **converges then** $\sum \boldsymbol{u_n}$**converges** and $\sum u_n \leq \sum v_n$.

(B) If $\boldsymbol{u_n \leq v_n}$, for all n≥1 and $\sum \boldsymbol{u_n}$ **diverges then** $\sum \boldsymbol{v_n}$**diverges** and $\sum u_n \leq \sum v_n$.

Test-2:

Let $\sum u_n$ and $\sum v_n$ be any two series of positive terms. If $\lim_{n\to\infty}\frac{u_n}{v_n} = l$.

Where **l is non zero and finite** then **both the series converges OR diverges together.**

Sr. No.	Question	Answer
1	In Comparision test, If $u_n \le v_n$, for all n≥1 and $\sum v_n$ converges then $\sum u_n$ is………	**Converges**
2	In Comparision test, If $u_n \le v_n$, for all n≥1 and $\sum u_n$ diverges then $\sum v_n$ is……	**Diverges**

❖ **Definition: P-Series OR Harmonic series**

The series $\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots\dots\dots\dots\dots + \frac{1}{n^p} + \dots\dots\dots\dots$ is called P-series or Harmonic series.

(ii) P-test:

The series $\sum \frac{1}{n^p}$ is (1) Convergent, if **p>1**

(2) Divergent, if **p≤1**

EXAMPLE-6:

Test the convergence OR divergence of the following series:

$$\frac{1}{1*2*3} + \frac{3}{2*3*4} + \frac{5}{3*4*5} + \frac{7}{4*5*6} + \dots\dots\dots\dots$$

SOLUTION:

Here, $u_n = \frac{(2n-1)}{n(n+1)(n+2)} = \frac{n\left(2-\frac{1}{n}\right)}{n^3\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} = \frac{\left(2-\frac{1}{n}\right)}{n^2\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$

We consider $\sum v_n = \sum \frac{1}{n^2} \Rightarrow$ **p=2>1**$\Rightarrow \sum \boldsymbol{v_n}$ **is convergent.**

Now,

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} n^2 * \frac{\left(2-\frac{1}{n}\right)}{n^2\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$$= \lim_{n\to\infty} \frac{\left(2-\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} = \frac{(2-0)}{(1+0)(1+0)} = \mathbf{2}$$

By Comparison test $\sum \boldsymbol{u_n}$ **is also convergent.**

EXAMPLE-7:

Test the convergence OR divergence of the following series:

$$\sum \sin\frac{1}{n}$$

SOLUTION:

Since $u_n = \sum \sin\frac{1}{n}$

We consider $\sum v_n = \sum \frac{1}{n}$, which is **divergent as p=1**

Now,

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}}=\lim_{\frac{1}{n}\to 0}\frac{\sin\frac{1}{n}}{\frac{1}{n}}=1 \quad \left(\text{Because,} \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1\right)$$

By Comparison test $\sum u_n$ **is also divergent.**

Sr. No.	Question	Answer
1	The series $\sum \frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ \dots\dots .+\frac{1}{n^p}+\ \dots\dots$ is called……….	**P-series**
2	P-series is also known as………	**Harmonic series**
3	Which test used for convergence of P-series?	**P-test**
4	Using P-test, we get p>1 then the series is…….	**Convergent**

EXERCISE – A

Test the convergence OR divergence of the following series:

1) $\frac{1}{1*2}+\frac{1}{2*3}+\frac{1}{3*4}+\ \dots\dots\dots\dots\dots..$
2) $\sum \cos\frac{1}{n}$
3) $\frac{1*2}{3^2*4^2}+\frac{3*4}{5^2*6^2}+\frac{5*6}{7^2*8^2}+\ \dots\dots\dots\dots$
4) $\sum\left(\sqrt{n+1}-\sqrt{n}\right)$
5) $\sum\frac{n+2}{n^3+1}$

(iii) D'Alemert's Ratio Test:

Let $\sum u_n$ be a series of positive terms and $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$.

(1) If **0≤l<1** then $\sum u_n$ **converges**

(2) If **l>1** then $\sum u_n$ **diverges**

(3) If **l=1** then from this test alone, we cannot draw any conclusion the convergence or divergence of $\sum u_n$.

EXAMPLE-8:

Test the convergence of the series $1 + \frac{2!}{2^2} + \frac{3!}{3^3} + \frac{4!}{4^4} + \dots \dots \dots \dots \dots$

SOLUTION:

Here, $u_n = \frac{n!}{n^n} \Rightarrow u_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$

Consider $\frac{u_{n+1}}{u_n} = \frac{(n+1)!}{(n+1)^{n+1}} * \frac{n^n}{n!} = \frac{n+1}{(n+1)^{n+1}} * n^n = \frac{n^n}{(n+1)^n} = \frac{n^n}{n^n\left(1+\frac{1}{n}\right)^n} = \frac{1}{\left(1+\frac{1}{n}\right)^n}$

$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$

$$\left[\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \;\&\; e > 2 \Rightarrow \frac{1}{e} < \frac{1}{2} < 1\right]$$

Hence, by D'Alembert's ratio test, $\sum u_n$ **is convergent.**

EXAMPLE-9:

Discuss the convergence of $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \dots \dots \dots \dots$

SOLUTION:

Here, $\boldsymbol{u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}} \Rightarrow u_{n+1} = \frac{x^{2n}}{(n+2)\sqrt{n+1}}}$

Now,

$$\frac{u_{n+1}}{u_n} = \frac{x^{2n}}{(n+2)\sqrt{n+1}} * \frac{(n+1)\sqrt{n}}{x^{2n-2}} = \frac{(n+1)}{(n+2)} * \sqrt{\frac{n}{n+1}} * x^2$$

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)}{(n+2)} * \sqrt{\frac{n}{n+1}} * x^2 = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{2}{n}\right)} * \sqrt{\frac{1}{1+\frac{1}{n}}} * x^2$$

$$= \frac{(1+0)}{(1+0)} * \sqrt{\frac{1}{1+0}} * x^2 = \boldsymbol{x^2}$$

$\therefore$ By ratio test, **if $x^2<1$, $\sum u_n$ is convergent** and **if $x^2>1$, $\sum u_n$ is divergent.**

When **$x^2=1$, ratio test fails**.

For $x^2=1$, then $u_n = \frac{1}{(n+1)\sqrt{n}} = \frac{1}{n^{\frac{3}{2}}\left(1+\frac{1}{n}\right)}$

We choose $\sum v_n = \sum \frac{1}{n^{\frac{3}{2}}}$, which **is convergent as $p = \frac{3}{2} > 1$**

Now,

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^{\frac{3}{2}}}{n^{\frac{3}{2}}\left(1+\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)} = \frac{1}{1+0} = \mathbf{1}$$

$\therefore$ By **comparison test, $\sum u_n$ is convergent.**

Hence, **if $x^2 \le 1$, $\sum u_n$ is convergent** and **if $x^2>1$, $\sum u_n$ is divergent.**

Sr. No.	Question	Answer
1	In ratio test we get l>1 then the series is…..	**Divergent**
2	In ratio test we get 0≤l<1 then the series is…..	**Convergent**

EXERCISE – B

Test the convergence of the following series:

1) $\frac{1}{1!}+\frac{3}{2!}+\frac{5}{3!}+ \dots\dots\dots\dots\dots$
2) $\sum_{n=1}^{\infty}\frac{n^2}{3^n}$
3) $2+\frac{3x}{2}+\frac{4x^2}{3}+\frac{5x^3}{4}+ \dots\dots\dots\dots$
4) $1+\frac{x}{2}+\frac{x^2}{5}+\frac{x^3}{10}+ \dots\dots\dots\dots+\frac{x^n}{n^2+1}+ \dots\dots\dots\dots$
5) $\frac{x}{1*3}+\frac{x^2}{3*5}+\frac{x^3}{5*7}+ \dots\dots\dots\dots\dots$

(iv) Raabe's Test:

Let $\sum u_n$ be a series of positive terms and $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)=l$.

(1) The series is **convergent**, **if l>1**

(2) The series is **divergent**, **if l<1**

(3) This test gives no information, if l=1

EXAMPLE-10:

Examine the convergence of $\frac{2}{3}+\frac{2*4}{3*5}+\frac{2*4*6}{3*5*7}+ \dots\dots\dots\dots\dots\dots\dots\dots$

SOLUTION:

Here, $\boldsymbol{u_n} = \frac{2*4*6*\dots\dots\dots*(2n)}{3*5*7*\dots\dots*(2n+1)} \Rightarrow \boldsymbol{u_{n+1}} = \frac{2*4*6*\dots\dots\dots*(2n)*(2n+2)}{3*5*7*\dots\dots*(2n+1)*(2n+3)}$

Now,

$$\frac{u_{n+1}}{u_n} = \frac{2*4*6*\dots\dots..*(2n)*(2n+2)}{3*5*7*\dots\dots*(2n+1)*(2n+3)} * \frac{3*5*7*\dots\dots\dots*(2n+1)}{2*4*6\dots\dots\dots\dots..*(2n)}$$

$$= \frac{2n+2}{2n+3}$$

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{2n+2}{2n+3} = \lim_{n\to\infty} \frac{n(2+\frac{2}{n})}{n(2+\frac{3}{n})} = \frac{(2+0)}{(2+0)} = \mathbf{1}$$

Therefore, D'Alembert's **ratio test fails.**

Now,

$$\frac{u_n}{u_{n+1}} - 1 = \frac{2n+3}{2n+2} - 1 = \frac{1}{2n+2}$$

$$\therefore \quad n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{n}{2n+2}$$

$$\therefore \quad \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} \frac{n}{2n+2} = \lim_{n\to\infty} \frac{n}{n\left(2+\frac{2}{n}\right)} = \frac{1}{(2+0)} = \frac{\mathbf{1}}{\mathbf{2}} < 1$$

By **Raabe's test, $\sum \boldsymbol{u_n}$ is divergent.**

EXAMPLE-11:

Discuss the convergence of $\sum \frac{1*3*5*\dots\dots\dots*(2n-1)}{2*4*6*\ \dots\dots\dots*(2n)} * x^n$

SOLUTION:

Here, $\boldsymbol{u_n} = \frac{1*3*5*\ldots\ldots*(2n-1)}{2*4*6*\ \ldots\ldots*(2n)} * \boldsymbol{x^n}$

$\Rightarrow \boldsymbol{u_{n+1}} = \frac{1*3*5*\ldots\ldots*(2n-1)*(2n+1)}{2*4*6*\ \ldots\ldots*(2n)*(2n+2)} * \boldsymbol{x^{n+1}}$

Now,

$$\frac{u_{n+1}}{u_n} = \frac{1*3*5*\ldots.*(2n-1)*(2n+1)}{2*4*6*\ \ldots.*(2n)*(2n+2)} * x^{n+1}\frac{2*4*6*.*(2n)}{1*3*5*\ldots.*(2n-1)*x^n}$$

$$= \frac{2n+1}{2n+2} * x$$

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{2n+1}{2n+2} * x = \lim_{n\to\infty} \frac{n(2+\frac{1}{n})}{n(2+\frac{2}{n})} * x = \boldsymbol{x}$$

$\therefore$ D'Alembert's ratio test, $\sum \boldsymbol{u_n}$ **is convergent if x<1** and **divergent if x>1.**

If x=1 then ratio test fails.

When x=1, $\frac{u_{n+1}}{u_n} = \frac{2n+1}{2n+2}$

We consider,

$$\frac{u_n}{u_{n+1}} - 1 = \frac{2n+2}{2n+1} - 1 = \frac{1}{2n+1}$$

$$\therefore\ n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{n}{2n+1}$$

$$\therefore\ \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{n}{n\left(2+\frac{1}{n}\right)} = \frac{1}{(2+0)} = \frac{\mathbf{1}}{\mathbf{2}} < 1$$

By **Raabe's test, $\sum u_n$ is divergent.**

Hence $\sum u_n$ **is convergent if x<1** and **divergent if $x \geq 1$.**

Sr. No.	Question	Answer
1	If the D'Alembert's ratio test fails, then what to do?	**Apply Raabe's test**
2	In Raabe's test if l=1 then the series is …	**Test gives no information**
3	In Raabe's test if l<1 then the series is ……….	**Divergent**

EXERCISE –C

Discuss the convergence of the following:

1) $\frac{1}{2} + \frac{1*3}{2*4} + \frac{1*3*5}{2*4*6} + \dots\dots\dots\dots\dots\dots$
2) $\sum \frac{4*7*10 \,\dots\dots\dots\dots (3n+1)}{1*2*3 \dots\dots\dots\dots n} * x^n$
3) $1 + \frac{3}{7}x + \frac{3*6}{7*10}x^2 + \frac{3*6*9}{7*10*13}x^3 + \dots\dots\dots\dots\dots\dots$
4) $\sum \frac{1^2*4^2*7^2 \,\dots\dots\dots\dots (3n-2)^2}{3^2*6^2*9^2 \,\dots\dots\dots\dots (3n)^2}$

❖ Definition: Absolutely convergent series

A series $\sum u_n$ is said to be absolutely convergent **if $\sum |u_n|$ is convergent.**

❖ Definition: Conditionally convergent series

A series $\sum u_n$ is said to be conditionally convergent **if $\sum u_n$ is convergent and $\sum |u_n|$ is divergent.**

Theorem – 4:

An absolutely convergent series is convergent.

EXAMPLE-12:

The series $1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \ldots\ldots\ldots\ldots\ldots\ldots$ is absolutely convergent as the series $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots\ldots\ldots\ldots\ldots\ldots$ is convergent as $r = \frac{1}{2} < 1$.

EXAMPLE-13:

The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots\ldots\ldots\ldots\ldots..$ is not absolutely convergent, since the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots\ldots\ldots\ldots\ldots..$ is divergent.

Also we know that the series $\mathbf{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots\ldots\ldots\ldots\ldots..}$ **is convergent.**

Hence, given series is **conditionally convergent but not a absolutely convergent**.

Sr. No.	Question	Answer
1	If $\sum |u_n|$ is convergent, then the series is called……….	**Absolutely convergent series**
2	If $\sum u_n$ is convergent and $\sum |u_n|$ is divergent, then the series is called…….	**Conditionally convergent series**

3	An absolutely convergent series is ………	**Convergent**

❖ Definition: Power series

A series of the **form** $\sum_{n=0}^{\infty} \boldsymbol{a_n x^n} = \boldsymbol{a_0 + a_1 x + a_2 x^2 +} \ldots\ldots\ldots\ldots . \boldsymbol{+ a_n x^n +} \ldots\ldots\ldots$where $a_0, a_1, a_2, \ldots\ldots\ldots\ldots\ldots$ are constants, is called a power series in x.

OR

A series of the form

$$\sum_{n=0}^{\infty} \boldsymbol{a_n (x-a)^n =}$$
$$\boldsymbol{a_0 + a_1 (x-a) + a_2 (x-a)^2 +} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . . \boldsymbol{+ a_n (x-a)^n +} \ldots\ldots\ldots$$

Is called a power series in (x-a).

Theorem-5:

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series and $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{R},\ R \in (-\infty, \infty)$

(i) If **|x|<R** then the series is **convergent.**

(ii) If **|x|>R** then the series is **divergent.**

(iii) If **|x|=R** then we can draw **no conclusion by this test**.

- We call number **R** in above theorem, the **radius of convergence** of $\sum_{n=0}^{\infty} a_n x^n$.

- The **collection of values of x** for which $\sum_{n=0}^{\infty} a_n x^n$ converges is called the **interval of convergence** OR the **range of convergence** of $\sum_{n=0}^{\infty} a_n x^n$.

- It seems that the interval of convergence takes one and only one of the following forms:

  [0, 0], (-R, R), [-R, R), (-R, R], [-R, R], (-∞, ∞)

EXAMPLE-14:

Find the radius of convergence for the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n+2}$$

SOLUTION:

Here, $\boldsymbol{u_n} = \frac{\boldsymbol{x^n}}{\boldsymbol{n+2}} \Rightarrow \boldsymbol{u_{n+1}} = \frac{\boldsymbol{x^{n+1}}}{\boldsymbol{n+3}}$

$$\therefore \frac{u_{n+1}}{u_n} = \left(\frac{x^{n+1}}{n+3}\right)\left(\frac{n+2}{x^n}\right) = \left(\frac{n+2}{n+3}\right)x$$

$$\therefore \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\left(\frac{n+2}{n+3}\right)x\right| = \lim_{n\to\infty}\left|\left(\frac{1+\frac{2}{n}}{1+\frac{3}{n}}\right)x\right| = |\boldsymbol{x}|$$

$\therefore$ By ratio test, the series is **converges if** $|\boldsymbol{x}| < 1$ and **diverges if** $|\boldsymbol{x}| > 1$.

$\therefore$ The **radius of convergence R=1**.

EXAMPLE-15:

Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

SOLUTION:

Since $\boldsymbol{u_n} = \frac{(-3)^n x^n}{\sqrt{n+1}} \Rightarrow \boldsymbol{u_{n+1}} = \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}}$

$$\therefore \frac{u_{n+1}}{u_n} = \left(\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}}\right)\left(\frac{\sqrt{n+1}}{(-3)^n x^n}\right) = -3x\sqrt{\frac{n+1}{n+2}}$$

$$\therefore \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|-3x\sqrt{\frac{n+1}{n+2}}\right| = \lim_{n\to\infty}\left|-3x\sqrt{\frac{1+\frac{1}{n}}{1+\frac{2}{n}}}\right| = \mathbf{3|x|}$$

$\therefore$ By ratio test, the series is **converges if** $\mathbf{3|x|} < 1 \Rightarrow |\boldsymbol{x}| < \frac{1}{3}$ and **diverges if** $\mathbf{3|x|} > 1 \Rightarrow |\boldsymbol{x}| > \frac{1}{3}$.

$\therefore$ The **radius of convergence is** $\boldsymbol{R} = \frac{1}{3}$ and the **interval of convergence is** $\left(-\frac{1}{3}, \frac{1}{3}\right)$

Sr. No.	Question	Answer
1	A series of the form $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots\dots\dots\dots + a_n x^n + \dots\dots\dots$ where $a_0, a_1, a_2, \dots\dots\dots\dots\dots$ are constants, is	**Power series**

	called…………	
2	If |x|<R then the series is ………….	**Convergent**
3	The collection of values of x for which $\sum_{n=0}^{\infty} a_n x^n$ converges is called…….	**Interval of convergence**
4	Interval of convergence is also known as…….	**Range of convergence**

EXERCISE –D

Find the radius of convergence and interval of convergence of the series:

1) $x + 2x^2 + 3x^3 + 4x^4 + \ \dots\dots\dots\dots\dots$
2) $2 - \frac{4}{5}x + \frac{6}{5^2}x^2 - \frac{8}{5^3}x^3 + \ \dots\dots\dots\dots$
3) $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$