SHREE H. N. SHUKLA GROUP OF COLLEGES

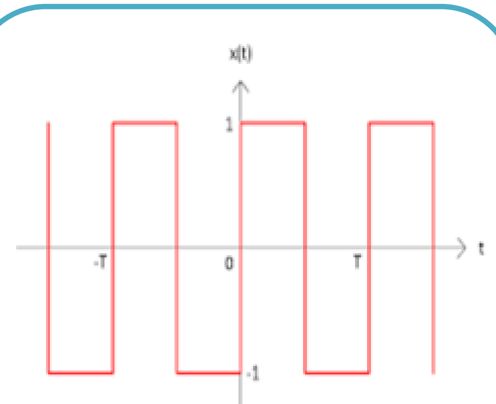
T.Y.B.Sc. SEM – V

Subject: Physics

Paper- 501

Unit -1

Fourier series



- Introduction
- Fourier series
- Co efficient of Fourier series
- Orthogonality condition
- Application of Fourier series
- Complex form of Fourier series
- Numerical

❖ TITLE JUSTIFICATION :

The title justified as that in 1822 a French mathematician Joseph Fourier invented a **Fourier series** Function. According to Fourier series if we have any **periodic signal** (**means** that signal repeat itself after some particular time) we can calculate their time period and frequency. In short, "Fourier series is a way of representing a periodic function as a (possibly infinite) sum of **sine** and **cosine** functions." Fourier series is a very powerful method to solve ordinary and partial differential equations.

❖ THEME :

In this chapter we will study about the **Fourier series** which is useful to find out the **frequency** of the **periodic function**. The Fourier series is a particular way of rewriting functions as a series of trigonometric functions like in the form of Sine & Cosine for the wave function. For functions that are **not periodic**, the Fourier series is replaced by the **Fourier transform**. Fourier series also use as the **Signal** processing, **Image** Processing, **Heat** distribution, **Wave** simplification, **Radiation** measurement etc….

❖ INTRODUCTION :

❖ Fourier Series:

- Any piecewise smooth function defined on a finite interval has a Fourier series expansion.
- In 1822 French mathematician J.B. FOURIER invented Fourier series. It is an infinite series representation of periodic function in terms of the trigonometric sine and cosine functions. Fourier series is a very powerful method to solve ordinary and partial differential equations.
- Fourier series is possible not only for continuous functions but also for periodic functions, functions which are discontinuous in their values and derivatives.
- **"A Fourier series is defined as an expansion of a periodic function or representation of a function in a series of sines and cosines,** "

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Here is the coefficients $\mathbf{a_0}$, $\mathbf{a_n}$ and $\mathbf{b_n}$ are the Fourier coefficients of f (x) defined as:

Watch now

Fourier coefficients of f(x), given by the **Euler formulas**

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

SR. NO	QUESTION	ANSWER
1	Expansion of periodic function as sine and cosine is known as___________.	Fourier series
2	What are the Fourier co efficient?	a_0, a_n and b_n
3	Write the value of a_0 co efficient ?	$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$
4	Write the value of a_n co efficient ?	$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$
5	Write the value of b_n co efficient ?	$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$
6	Write the Fourier series equation in terms of sine & cosine.	$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$

❖ Fourier Series Co - efficient :

Let us further assume that f(x) can be represented by a trigonometric series,

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \qquad -\pi \leqslant x \leqslant \pi$$

.... (A)

- FOR DETERMINATION OF a_0 :

if we integrate both side of equation (A) :

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} a_0\,dx + \int_{-\pi}^{\pi}\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\,dx$$

$$= 2\pi a_0 + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi}\cos nx\,dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi}\sin nx\,dx$$

but

$$\int_{-\pi}^{\pi}\cos nx\,dx = \frac{1}{n}\sin nx\Big]_{-\pi}^{\pi} = \frac{1}{n}[\sin n\pi - \sin(-n\pi)] = 0$$

because n is an integer. Similarly,

$$\int_{-\pi}^{\pi} \sin nx\, dx = 0. \text{ So}$$

$$\int_{-\pi}^{\pi} f(x)\, dx = 2\pi a_0$$

and solving for a_0 gives

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

..... (1)

- FOR DETERMINATION OF a_n :

For determine we have multiply equation A with cos mx and integrate with limits.

$$\int_{-\pi}^{\pi} f(x)\cos mx\, dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right]\cos mx\, dx$$

$$= a_0\int_{-\pi}^{\pi}\cos mx\, dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi}\cos nx\cos mx\, dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi}\sin nx \cos mx\, dx$$

For using orthogonality function for above equation , the first and third term will be zero for above equation. The only nonzero term is $a_m\pi$ and we get ,

$$\int_{-\pi}^{\pi} f(x)\cos mx\, dx = a_m\pi$$

Solving for am, and then replacing m by n , we have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx \qquad n = 1, 2, 3, \ldots$$

.... (2)

- FOR DETERMINATION OF b_n :

Similarly, if we multiply both sides of Equation **A** with **sin mx** by and integrate them, we get

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx \qquad n = 1, 2, 3, \ldots$$

.... (3)

So we get below result :

Fourier coefficients of f(x), given by the **Euler formulas**

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

❖ Orthogonality condition :

- $\mathcal{F}\{Trigonometric\ Poly.\} = Itself$
- $\int_{-\pi}^{\pi} \sin nx\ \sin mx\ dx = \begin{cases} 0 & : n \neq m \\ \pi & : n = m \end{cases}$
- $\int_{-\pi}^{\pi} \sin nx\ \cos mx\ dx = 0\ always$
- $\int_{-\pi}^{\pi} \cos nx\ \cos mx\ dx = \begin{cases} 0 & : n \neq m \\ \pi & : n = m \end{cases}$

SR.NO	QUESTION	ANSWER
1	cos n**π** =	$(-1)^n$
2	$\int_{-\pi}^{\pi} \sin nx \cos nx\, dx =$	**0**
3	$\int_{-\pi}^{\pi} \cos nx \cos mx\, dx =$	**π**

Problems on Fourier Series

1) Find the Fourier series to represent $f(x) = x^2$ in the interval $(0, 2\pi)$.

Sol: We know that, the Fourier series of $f(x)$ defined in the interval $(0, 2\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where, $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\, dx$$

Here, $f(x) = x^2$

Now, $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx = \frac{1}{\pi}\int_0^{2\pi} x^2\, dx$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi} = \frac{1}{3\pi}[(2\pi)^3 - 0] = \frac{8}{3}\pi^2$$

$$\Rightarrow \boxed{a_0 = \frac{8}{3}\pi^2}$$

Again, $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\, dx = \frac{1}{\pi}\int_0^{2\pi} \underbrace{x^2}_{u}\, \underbrace{\cos nx}_{v}\, dx$

$$= \frac{1}{\pi}\left[x^2 \int \cos nx\, dx - \left\{\int \frac{d}{dx}(x^2)\left(\int \cos nx\, dx\right)dx\right\}\right]$$

$$\left[\because \int uv\, dx = u\int v\, dx - \left\{\int \frac{du}{dx}.\left(\int v\, dx\right)dx\right\}\right]$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - \left\{\int 2x\left(\frac{\sin nx}{n}\right)dx\right\}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - \frac{2}{n}\left\{\int \underbrace{x}_{u}\, \underbrace{\sin nx}_{v}\, dx\right\}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - \frac{2}{n}\left(-x\frac{\cos nx}{n} + \int 1.\frac{\cos nx}{n}dx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - \frac{2}{n}\left(-x\frac{\cos nx}{n} + \frac{1}{n}\int \cos nx\, dx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - \frac{2}{n}\left(-x\frac{\cos nx}{n} + \frac{1}{n}\frac{\sin nx}{n}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) + \frac{2}{n^2}x \cos nx - \frac{2}{n^3}\sin nx\right]_0^{2\pi}$$

$$= \frac{4}{n^2} \quad \left[\because \begin{array}{l}\cos 2n\pi = 1\\ \sin 2n\pi = 0\end{array}\right]$$

$$\Rightarrow \boxed{a_n = \frac{4}{n^2}}$$

Again, $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} \underbrace{x^2}_{u}\,\underbrace{\sin nx}_{v}\,dx$

$$= \frac{1}{\pi}\left[x^2\int \sin nx\,dx - \left\{\int \frac{d}{dx}(x^2)\left(\int \sin nx\,dx\right)dx\right\}\right]$$

$$\left[\because \int uv\,dx = u\int v\,dx - \left\{\int \frac{du}{dx}.\left(\int v\,dx\right)dx\right\}\right]$$

$$= \frac{1}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - \left\{\int 2x\left(-\frac{\cos nx}{n}\right)dx\right\}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-x^2\left(\frac{\cos nx}{n}\right) + \frac{2}{n}\left\{\int \underbrace{x}_{u}\,\underbrace{\cos nx}_{v}\,dx\right\}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-x^2\left(\frac{\cos nx}{n}\right) + \frac{2}{n}\left(x\frac{\sin nx}{n} + \int 1.\frac{\sin nx}{n}dx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-x^2\left(\frac{\cos nx}{n}\right) + \frac{2}{n}\left(x\frac{\sin nx}{n} + \frac{1}{n}\int \sin nx\,dx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-x^2\left(\frac{\cos nx}{n}\right) + \frac{2}{n}\left(x\frac{\sin nx}{n} + \frac{1}{n}\frac{\cos nx}{n}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-x^2\left(\frac{\cos nx}{n}\right) + \frac{2}{n^2}x\sin nx + \frac{2}{n^3}\cos nx\right]_0^{2\pi}$$

$$= -\frac{4\pi}{n} \qquad \left[\because \begin{matrix}\cos 2n\pi = 1\\ \sin 2n\pi = 0\end{matrix}\right]$$

$$\Rightarrow \boxed{b_n = -\frac{4\pi}{n}}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$\therefore f(x) = x^2 = \frac{\frac{8\pi^2}{3}}{2} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$$

$$\Rightarrow x^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$$

This is the Fourier series for the function $f(x) = x^2$

Hence the result

2) Find the Fourier series of the periodic function defined as $f(x) = \begin{cases} -\pi & ; -\pi < x < 0 \\ x & ; \quad 0 < x < \pi \end{cases}$

Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$

Sol: We know that, the Fourier series of $f(x)$ defined in the interval $(-\pi, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$

Here, $f(x) = \begin{cases} -\pi & ; -\pi < x < 0 \\ x & ; \quad 0 < x < \pi \end{cases}$

Now, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\, dx + \int_{0}^{\pi} f(x)\, dx\right]$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\, dx + \int_{0}^{\pi} x\, dx\right]$$

$$= \frac{1}{\pi}\left[(-\pi)\int_{-\pi}^{0} dx + \int_{0}^{\pi} x\, dx\right]$$

$$= \frac{1}{\pi}\left[(-\pi)[x]_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_0^{\pi}\right] = \frac{1}{\pi}\left[(-\pi)(\pi) + \frac{\pi^2}{2}\right]$$

$$= \frac{1}{\pi}\left[-\pi^2 + \frac{\pi^2}{2}\right] = -\frac{\pi}{2}$$

$$\Longrightarrow \boxed{a_0 = -\frac{\pi}{2}}$$

Also, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x) \cos nx\, dx + \int_{0}^{\pi} f(x) \cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi) \cos nx\, dx + \int_{0}^{\pi} x \cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\int_{-\pi}^{0}(\cos nx)\, dx + \int_{0}^{\pi} x \cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left\{x\left(\frac{\sin nx}{n}\right) - \int 1\left(\frac{\sin nx}{n}\right) dx\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}(\sin nx)_{-\pi}^{0} + \left\{\frac{x \sin nx}{n} - \frac{1}{n}\int \sin nx\, dx\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}(\sin nx)_{-\pi}^{0} + \left\{\frac{x \sin nx}{n} - \frac{1}{n}\frac{(-\cos nx)}{n}\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}(\sin nx)_{-\pi}^{0} + \left\{\frac{x\sin nx}{n} + \frac{1}{n^2}\cos nx\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}[0-\sin(-n\pi)] + \left\{\left(\frac{\pi\sin n\pi}{n} + \frac{1}{n^2}\cos n\pi\right) - \left(0 + \frac{1}{n^2}\cos n0\right)\right\}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\sin n\pi + \left\{\left(\frac{\pi\sin n\pi}{n} + \frac{1}{n^2}\cos n\pi\right) - \frac{1}{n^2}.1\right\}\right] \qquad \left[\because \begin{matrix}\cos(-\theta) = \cos\theta \\ \sin(-\theta) = \sin\theta\end{matrix}\right]$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[-\frac{\pi}{n}(0) + \left\{\left(\frac{\pi(0)}{n} + \frac{1}{n^2}(-1)^n\right) - \frac{1}{n^2}\right\}\right] \qquad \left[\because \begin{matrix}\sin n\pi = 0 \\ \& \\ \cos n\pi = (-1)^n\end{matrix}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} + \frac{1}{n^2}\right] = \frac{1}{\pi n^2}[(-1)^n - 1]$$

$$\Rightarrow \boxed{a_n = \frac{1}{\pi n^2}[(-1)^n - 1]}$$

Again, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_0^{\pi} f(x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin nx\,dx + \int_0^{\pi} x\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\int_{-\pi}^{0}(\sin nx)\,dx + \int_0^{\pi} x\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left(-\frac{\cos nx}{n}\right)_{-\pi}^{0} + \left\{x\left(-\frac{\cos nx}{n}\right) - \int 1\left(-\frac{\cos nx}{n}\right)dx\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(\cos nx)_{-\pi}^{0} + \left\{-\frac{x\cos nx}{n} + \frac{1}{n}\int\cos nx\,dx\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(\cos nx)_{-\pi}^{0} + \left\{-\frac{x\cos nx}{n} + \frac{1}{n}\frac{(\sin nx)}{n}\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(\cos nx)_{-\pi}^{0} + \left\{-\frac{x\cos nx}{n} + \frac{1}{n^2}\sin nx\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}[1-\cos(-n\pi)] + \left\{\left(-\frac{\pi\cos n\pi}{n} + \frac{1}{n^2}\sin n\pi\right) - \left(-0 + \frac{1}{n^2}\sin n0\right)\right\}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}[1-\cos n\pi] - \frac{\pi\cos n\pi}{n}\right] = \frac{1}{n}(1 - 2\cos n\pi)$$

$$\left[\because \begin{matrix}\cos(-\theta) = \cos\theta \\ \sin(-\theta) = \sin\theta\end{matrix} \;\&\; \begin{matrix}\sin n\pi = 0 \\ \& \\ \cos n\pi = (-1)^n\end{matrix}\right]$$

$$\Rightarrow \boxed{b_n = \frac{1}{n}(1 - 2\cos n\pi)}$$

Hence, the Fourier series for given $f(x)$ is given by

$$f(x) = \frac{-\frac{\pi}{2}}{2} + \sum_{n=1}^{\infty}\left(\frac{1}{\pi n^2}[(-1)^n - 1]\cos nx + \frac{1}{n}(1 - 2\cos n\pi)\sin nx\right)$$

$$\Rightarrow f(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{1}{\pi n^2}[(-1)^n - 1]\cos nx + \frac{1}{n}(1 - 2\cos n\pi)\sin nx\right)$$

$$\Rightarrow f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ \dots\right) + \left(3\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \ \dots\right)$$

Deduction: Put $x = 0$ in the above function $f(x)$, we get

$$f(0) = -\frac{\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ \dots\right)$$

Since, $f(x)$ is discontinuous at $x = 0$, $f(0-0) = -\pi$, $f(0+0) = 0$

$$\Rightarrow f(0) = \frac{1}{2}[f(0-0) + f(0+0)]$$

$$\Rightarrow f(0) = \frac{1}{2}(-\pi) = -\frac{\pi}{2}$$

Hence, $f(0) = -\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ \dots\right)$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ \dots \ = \frac{\pi^2}{8}$$

Hence the result

3) Expand the function $f(x) = x^2$ as Fourier series in $[-\pi, \pi]$.

Hence deduce that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ \dots = \frac{\pi^2}{6}$

Sol: We know that, the Fourier series of $f(x)$ defined in the interval $(-\pi, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

Here, $f(x) = x^2$

Now, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2}{\pi}\int_{0}^{\pi} x^2\,dx$

$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2\pi^2}{3}$

$$\Longrightarrow \boxed{a_0 = \frac{2\pi^2}{3}}$$

Again, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos nx\,dx \qquad \left[\because f(x)\ is\ even \Longrightarrow \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx\right]$$

$$= \frac{2}{\pi}\left[\frac{x^2 \sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2x\sin nx}{n^3}\right]_0^{\pi} = \frac{4}{n^2}(-1)^n$$

$$\Longrightarrow \boxed{a_n = \frac{4}{n^2}(-1)^n}$$

Again, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx\,dx$$

$$= 0 \qquad \left[\because f(x)\ is\ odd \Longrightarrow \int_{-a}^{a} f(x)dx = 0\right]$$

Hence, the Fourier series for given $f(x)$ is given by

$$f(x) = x^2 = \frac{\left(\frac{2\pi^2}{3}\right)}{2} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n \cos nx$$

$$\Longrightarrow x^2 = \frac{\pi^2}{3} + 4\left(-\cos x + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \frac{\cos 4x}{4^2} - \ \dots\right)$$

Deduction: Put $x = \pi$ in the above equation, we get

$$\Longrightarrow \pi^2 = \frac{\pi^2}{3} + 4\left(-\cos\pi + \frac{\cos 2\pi}{2^2} - \frac{\cos 3\pi}{3^2} + \frac{\cos 4\pi}{4^2} - \ \dots\right)$$

$$\Longrightarrow \pi^2 - \frac{\pi^2}{3} = 4\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ \dots\right)$$

$$\Longrightarrow \frac{2\pi^2}{3} = 4\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ \dots\right)$$

$$\Longrightarrow \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ \dots$$

Hence the Result

SR.NO	QUESTION	ANSWER
1	$f(x)=x+x^2$ for $-\pi<x<\pi$	$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots..$
2	$f(x)=x+x^2$ for $-\pi<x<\pi$	$\frac{\pi^2}{6}$
3	$\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots..$	$\sum_{n=1}^{\infty}\frac{1}{n2}$
4	$f(x)=\begin{cases}-\pi & -\pi<x<0\\ x, & 0<x<\pi\end{cases}$	$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots...$
5	$f(x)=\begin{cases}-\pi & -\pi<x<0\\ x, & 0<x<\pi\end{cases}$	$=\frac{\pi^2}{8}$
6	$f(x)=x^2, \quad -\pi\le x\le\pi$	$\frac{1}{1^2} \quad \frac{1}{2^2} \quad \frac{1}{3^2} \quad \frac{1}{4^2} \ldots.$
7	$f(x)=x^2, \quad -\pi\le x\le\pi$	$=\frac{\pi^2}{12}$

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