The Sharpe Index Model

Need for Sharpe Model

* In Markowitz model a number of co-variances have to be estimated.
* If a financial institution buys 150 stocks, it has to estimate 11,175 *i.e.,* $(N^2 - N)/2$ correlation co-efficients.
* Sharpe assumed that the return of a security is linearly related to a single index like the market index.

Single Index Model

- Casual observation of the stock prices over a period of time reveals that most of the stock prices move with the market index.

- When the Sensex increases, stock prices also tend to increase and vice – versa.

- This indicates that some underlying factors affect the market index as well as the stock prices.

Stock prices are related to the market index and this relationship could be used to estimate the return of stock.

$$R_i = \alpha_i + \beta_i R_m + e_i$$

where R_i — expected return on security i

α_i — intercept of the straight line or alpha co-efficient

β_i — slope of straight line or beta co-efficient

R_m — the rate of return on market index

e_i — error term

Risk

- Systematic risk $\quad=\beta_i^2 \times$ variance of market index
  $$= \beta_i^2 \sigma_m^2$$

- Unsystematic risk = Total variance – Systematic risk
  $$e_i^2 \quad = \sigma_i^2 - \text{Systematic risk}$$

- Thus the total risk = Systematic risk + Unsystematic risk
  $$= \beta_i^2 \sigma_m^2 + e_i^2$$

Portfolio Variance

$$\sigma_p^2 = \left[\left(\sum_{i=1}^{N} x_i \, \beta_i\right)^2 \sigma_m^2\right] + \left[\sum_{i=1}^{N} x_i^2 \, e_i^2\right]$$

where

σ_p^2 = variance of portfolio

σ_m^2 = expected variance of market index

e_i^2 = Unsystematic risk

x_i = the portion of stock *i* in the portfolio

Example

- The following details are given for x and y companies' stocks and the Sensex for a period of one year. Calculate the systematic and unsystematic risk for the companies stock. If equal amount of money is allocated for the stocks , then what would be the portfolio risk ?

	X stock	Y stock	Sensex
Average return	0.15	0.25	0.06
Variance of return	6.30	5.86	2.25
Beta	0.71	0.27	

Company X

- Systematic risk $= \beta_i^2 \times$ variance of market index

$$= \beta_i^2 \sigma_m^2 = (0.71)^2 \times 2.25 = 1.134$$

- Unsystematic risk = Total variance – Systematic risk

$$e_i^2 = \sigma_i^2 - \text{Systematic risk} = 6.3 - 1.134 = 5.166$$

- Total risk = Systematic risk + Unsystematic risk

$$= \beta_i^2 \sigma_m^2 + e_i^2 = 1.134 + 5.166 = 6.3$$

Company Y

- Systematic risk $= \beta_i^2 \times$ variance of market index

$$= \beta_i^2 \sigma_m^2 = (0.27)^2 \times 2.25 = 0.1640$$

- Unsystematic risk= Total variance – Systematic risk

$$e_i^2 = \sigma_i^2 - \text{Systematic risk} = 5.86 - 1.134 = 5.166$$

- σ_p^2 = [(.5 x .71 + .5 x .27)^2 2.25] + [(.5)^2 (5.166) + (.5)^2(5.696)]
- = [(.355 + .135)2 2.25] + [(1.292 + 1.424)]
- = 0.540 + 2.716
- = 3.256

Corner portfolio

- The entry or exit of a new stock in the portfolio generates a series of corner portfolio.

- In an one stock portfolio, it itself is the corner portfolio .

- In a two stock portfolio, the minimum risk and the lowest return would be the corner portfolio.

- As the number of stocks increases in a portfolio, the corner portfolio would be the one with lowest return and risk combination.

Expected Return of Portfolio

$$R_p = \sum_{i=1}^{N} x_i(\alpha_i + \beta_i R_m)$$

- For each security α_i and β_i should be estimated

- Portfolio return is the weighted average of the estimated return for each security in the portfolio.

- The weights are the respective stocks' proportions in the portfolio.

Portfolio Beta

$$\beta_p = \sum_{i=1}^{N} x_i \beta_i$$

A portfolio's beta value is the weighted average of the beta values of its component stocks using relative share of them in the portfolio as weights.

β_p is the portfolio beta.

Sharpe's optimal portfolio

$$\frac{R_i - R_f}{\beta_i}$$

❖ The selection of any stock is directly related to its excess return to beta ratio.

■ where R_i = the expected return on stock i

R_f = the return on a risk less asset
β_i = Systematic risk

Optimal Portfolio

❖ The steps for finding out the stocks to be included in the optimal portfolio are as:

➢ Find out the "excess return to beta" ratio for each stock under consideration

➢ Rank them from the highest to the lowest

➢ Proceed to calculate C_i for all the stocks according to the ranked order using the following formula

$$C_i \dfrac{\sigma_m^2 \displaystyle\sum_{i=1}^{N} \dfrac{(R_i - R_f)\beta_i}{\sigma_{ei}^2}}{1 + \sigma_m^2 \displaystyle\sum_{i=1}^{N} \dfrac{\beta_i^2}{\sigma_{ei}^2}}$$

σ_m^2 = variance of the market index

σ_{ei}^2 = stock's unsystematic risk

Cut-off point

➤ The cumulated values of C_i start declining after a particular C_i and that point is taken as the cut-off point and that stock ratio is the cut-off ratio C.

➤ σ_{ei}^2 - Unsystematic risk

$(R_i - R_f) / \beta_i$ - Excess return **to Beta**

Example

- Data for finding out the optimal portfolio are given below

Security number	Mean return R_i	Excess return $R_i - R_f$	Beta β	σ_{ei}^2	$(R_i - R_f)/\beta_i$
1	19	14	1.0	20	14
2	23	18	1.5	30	12
3	11	6	0.5	10	12
4	25	20	2.0	40	10
5	13	8	1.0	20	8
6	9	4	0.5	50	8
7	14	9	1.5	30	6

- **The riskless rate of intrest is 5% and the market variance is 10. Determine the cut – off point .**

- C1 = (10 x .7)/ [1 + (10 x .05)] =4.67
- C2 = (10 x 1.6)/ [1 + (10 x .125)] =7.11
- C3 = (10 x 1.9)/ [1 + (10 x .15)] =7.6
- C4 = (10 x 2.9)/ [1 + (10 x .25)] = 8.29* Cut-off point
- C5 = (10 x 3.3)/ [1 + (10 x .3)] = 8.25
- C6 = (10 x 3.34)/ [1 +(10 x .305)] = 8.25
- C7= (10 x 3.79)/ [1 + (10 x .38)] = 7.90

- The highest C_i value is taken as the cut-off point i.e C^*.

- The stocks ranked above C^* have high excess returns to beta than the cut off C_i and all the stocks ranked below C^* have low excess return to beta.

- Here the cut off rate is 8.29.

- Hence the first four securities are selected.

- If the number of stocks is larger there is no need to calculate c_i values for all the stocks after the ranking has been done.

- It can be calculated until the C^* value is found and after calculating for one or two stocks below it, the calculations can be terminated.

Construction of the optimal portfolio

- After determining the securities to be selected, the portfolio manager should find out how much should be invested in each security.The percentage of funds to be invested in each security can be estimated as follows .

- $Z_i = (\beta_i / \sigma^2_{ei}) \times [(R_i - R_f / \beta_i) - C]$

- $Z_i = (\beta_i / \sigma^2_{ei}) \times [(R_i - R_f / \beta_i) - C]$
- $Z_1 = (1/20) \times (14 - 8.29)$ = 0.285
- $Z_2 = (1.5 / 30) \times (12 - 8.29)$ = 0.186
- $Z_3 = (0.5 / 10) \times (12 - 8.29)$ = 0.186
- $Z_4 = (2 / 40) \times (10 - 8.29)$ = 0.086
- Total   = .743

- $X_1 = 0.285 / .743 = 0.38$
- $X_2 = 0.186 / .743 = 0.25$
- $X_3 = 0.186 / .743 = 0.25$
- $X_4 = 0.086 / .743 = 0.12$

- So the largest investment should be made in security 1 (0.38%) and the smallest in security 4 (0.12%).

- The characteristics of a stock that make it desirable can be determined before the calculations of an optimal portfolio is begun.

- The desirability of any stock is solely a function of its excess return to beta ratio.