Shree H.N. Shukla College of Science

M. Sc (Mathematics) (Sem-4)

Preliminary Exam

MATH.EMT-4001: Linear Algebra

[Time: 2.5 Hours] [Total Marks: 70]

Instructions : (1) All questions are compulsory.

(2) There are 5 questions.

(3) Figures on right side indicate full marks .

1. <u>Attempt any seven</u> : **7X2=14**

(1) Let T,S $\in A_F(V)$. Im standard notation verify that $r(ST) \leq r(T)$.

(2) Let A,B$\in F_n$ .In standard notation verify that tr(A+B)= tr(A)+tr(B), where tr : $F_n \rightarrow F$ and it is a trace of a matrix .

(3)Define an algebra over a field F.Also give an example of an algebra over the field F.

(4)Define similar linear transformations and similar matrices for a finite dimensional vector space V over F .

(5)Let $n \geq 1$ and A$\in F_n$. Define tr(A) .Verify that the map tr:Fn $\rightarrow$ Fis linear .

(6)Let (V, < >) be an inner product space over $\mathbb{C}$.If a linear transformation T : V$\rightarrow$ V is unitary , then prove that $T^*T = Id_V$.

(7)Let T$\in A_F(V)$.When is a subspace W of V is said to be invariant under T ? Verify that $T^2(V)$ is a subspace of V and is invariant under T.

(8)Let V be a vector space over R.Let f : VxV$\rightarrow$ R be bilinear .When is f said to be skew-symmetric ? If f is skew-symmetric, then show that f(v,v) =0 for any v$\in$ V .

(9) When is an element $\lambda \in F$ said to be characteristic root of $T \in A_F(V)$?
Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be given by T(1,0) =(1,1) and T(0,1) =(1,-1) .
Determine the characteristic root of T .

(10) Let $n \geq 1$ and $A \in F_n$ .Define the secular equation of A.Let $A \in F_3$ be any diagonal matrix .Determine the secular equation of A .

2 <u>Answer any two</u> **2X7=14**

(1) If n_1 is the index of nilpotence of a nilpotent $T \in A_F(V)$ and if $v \in V$ is such that $T^{n_1-1}(v) \neq 0$, then prove that $\{v , T(v), \ldots, T^{n_1-1}(v)\}$ is linearly independent over F.

(2) Let V is a n- dimentional vector space over a field F , then prove that $A_F(V)$ and F_n are isomorphic as algebras over F.

(3) Prove that any $T \in A_F(V)$ satisfies its characteristic polynomial.

(4) Let $T_1, T_2 \in A_F(V)$ and they both have same invariants. Prove that they are similar .

3 <u>Attempt the following</u> : **1X14 =14**

(1) Let $tr(T^k) = 0$, $\forall$ $k \in \mathbb{N}$. Prove that T is nilpotent.

(2) Let $A, B \in F_n$. Prove that det (AB) = det(A) det(B) .

(3) Let $T_1, T_2 \in A_F(V)$ and they both have same invariants. Prove that they are similar .

(4) Let $A \in F_n$ and suppose that K is the splitting field of minimal polynomial of A over F.Show that there is an invertible matrix $C \in K_n$ such that CAC^{-1} is in Jordan form.

4 <u>Attempt the following</u> : **2X7 = 14**

(1) Let $T \in A_F(V)$.If V is cyclic relative to T, then prove that there exist a basis B of V over F such that the matrix of T in B is C(p(x)) , where p(x) is the minimal polynomial of T over F.

(2) Let A∈ $\mathbb{C}_n$ be Hermitian .Show that any characteristic root of A is real.

(3) Let A∈ F_n .Prove that the interchanging two rows of A change the sign of its determinant.

(4) Let A,B∈ F_n . Prove that tr(AB)=tr(BA) .

5 <u>Attempt the following</u> : **2X7 =14**

(1)Let (V, < >) be a finite dimensional inner product space over ℂ .Let T∈A_F(V) .Then given v∈ V, prove that there exist an element w∈ V, depending on v and T such that <T(u) ,v> = < u, w > , for all v ∈V .

(2)Let F be a field of characteristic 0.If T,S ∈A_F(V) are such that ST-TS commutes with S, then prove that ST-TS is nilpotent .

(3)Let f: V x V→ F be a bilinear form on an n-dimentional vector space V over F .If B , B' are any two basis of V over F ,then prove that there exist an invertible matrix C∈ F_n such that $[f]_{B'}$ = C $[f]_B$C'.

(4)Let (V, < >) be a finite dimensional inner product space over ℂ. If N∈ A_C is normal ,then prove that there exist an orthonormal basis of V consisting of characteristic vectors of N, in which the matrix of N is diagonal

Best Of Luck