SHREE H.N.SHUKLA INSTITUTE OF PHARMACEUTICAL EDUCATION AND RESEARCH

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SUBJECT NAME: REMEDIAL MATHEMATICS

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UNIT 2:

MATRICES AND DETERMINANT

Topic

Matrices and Determinant

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- Types of matrices
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- Determinants
- Properties of determinants
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Introduction Matrices

Consider the System of Equations:

- $5x + 2y = 1,\ 3x - 4y = 6$
  Here, all x and y are unknown and the coefficients of unknown are all constants.
- An arrangement of coefficient of unknown into rows and columns is known as an Array.
- We obtain a rectangular array of the given system of equations as $\begin{pmatrix} 5 & 2 \\ 3 & -4 \end{pmatrix}$ or $\begin{bmatrix} 5 & 2 \\ 3 & -4 \end{bmatrix}$.
- This systematic arrangement of numbers in rectangular array is known as a matrix.
- In any matrix, the vertical line is known as column and the horizontal line is known as row.
- The numbers 5, 2, 3 etc... Are called the elements or entries of the matrix.
- In general, let the system of equations,
  $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ .... + a_{1n}x_n = b_1$
  $a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ .... + a_{2n}x_n = b_2$
  $a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ .... + a_{3n}x_n = b_3$
  ....
  ....
  $a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ .... + a_{mn}x_n = b_m$
- In matrix form,
  $AX = B$

  Where, $A = Thecoefficientmatrix = \left[a_{ij}\right]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & \cdots & \cdots & a_{mn} \end{bmatrix}$

  $B = Columnmatrix = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ \cdots \\ b_m \end{bmatrix}$ And $X = Columnmatrix = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ \cdots \\ x_n \end{bmatrix}$

  Where, $i = 1,2,3, ..., m \,\&\, j = 1,2,3, ..., n$
- Also, $(m \times n)$ is called the order of the matrix, read as ‘m by n’ matrix.
- In short, a well defined collection of numbers (real or complex) or objects in a rectangular array is called a matrix.

Types of matrices

1) Square Matrix:

- An m×n matrix for which the number of rows is equal to the number of column is known as a square matrix.
- Let $A = [a_{ij}]_{n\times n}$ be a square matrix of order n, where a_{ij} are the elements of the square matrix.
- **Example:**

The matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 8 & 9 & 4 \end{bmatrix}_{3\times 3}$ is a square matrix of order 3.

2) Unit (Identity) Matrix:

- An n×n matrix square matrix is said to be an unit matrix, if all diagonal elements are unity that means 1 and all non-diagonal elements are equal to zero.
- It is denoted by$I$.
- Mathematically, $a_{ij} = 1, if i = j and a_{ij} = 0, if i \neq j$
- **Example:**

$I_1 = [1]_{1\times 1}$, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$ And so on.

3) Null (zero) Matrix:

- An m×n matrix is said to be a null or zero matrix, if all elements a_{ij} are zero.
- It is denoted by O.
- **Example:**

$O_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$, $O_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$

4) Row Matrix:

- Any 1×n matrix, which has only one row and n column, is called a row matrix.
- **Example:**

$A = [2 \quad 0 \quad -1 \quad 4]_{1\times 4}$, $B = [1 \quad 3 \quad 5]_{1\times 3}$

5) Column Matrix:

- Any m×1 matrix, which has m rows and only one column, is called a column matrix.
- **Example:**

$$A = \begin{bmatrix} a \\ b \\ c \end{bmatrix}_{3\times 1} \quad , \quad X = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}_{3\times 1}$$

6) Diagonal Matrix:

- Any square matrix with non zero diagonal elements is called a diagonal matrix, that means $a_{ij} = 0\ for i \neq j and a_{ij} = non - zero number$
- **Example:**

$$A = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix}_{3\times 3} \quad , \quad B = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 2 \end{bmatrix}_{3\times 3}$$

7) Equality of two matrices:

- Consider $A = [a_{ij}]$ and $B = [b_{ij}]$ are two matrices. If matrices A and B are equal, then
  (i) They have same size.
  (ii) All the elements in the corresponding places of the two matrices are the same, that means $a_{ij} = b_{ij} for all i and j$

  Otherwise, we said two matrices A and B are not equal.

Operation on Matrices

1) Addition of matrices:

- Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ be two matrices of the same size, then their addition are denoted by (A+B) and obtained matrix is also of order m×n.
- i.e. $A + B = [a_{ij} + b_{ij}]_{m\times n}$
- **Example:**

  If $A = \begin{bmatrix} 1 & 5 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then $A + B = \begin{bmatrix} 1 & 5 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+1 & 5+0 \\ -1+0 & 3+1 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ -1 & 4 \end{bmatrix}$

2) Subtraction of Matrices:

- Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ be two matrices of the same order.
- Then $A - B = [a_{ij}]_{m\times n} - [b_{ij}]_{m\times n} = [a_{ij} - b_{ij}]_{m\times n}$
- **Example:**

  If $A = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 6 \\ -5 & 2 \end{bmatrix}$ then

  $$A + B = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} - \begin{bmatrix} 7 & 6 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 1-7 & 2-6 \\ 4-(-5) & -3-2 \end{bmatrix} = \begin{bmatrix} -6 & -4 \\ 9 & -5 \end{bmatrix}$$

3) Multiplication of Matrices:

- Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{jk}]_{n\times p}$ be two matrices such that the number of columns of matrix A is equal to the number of rows of matrix B, then the obtain matrix will be order (m×p).
- $AB = [a_{ij}]_{m\times n} \cdot [b_{jk}]_{n\times p} = [c_{ik}]_{m\times p} = C,$

  $where c_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk}$ (This summation is w.r.t. repeated indices j)

- **Example:**

Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 2 & -1 & 4 \end{bmatrix}_{3\times 3}$ and $B = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 5 & 1 \end{bmatrix}_{3\times 2}$

Since the matrix A is of order 3×3 and the matrix B is of order 3×2.

Here the number of column in A is equal to the number of rows in B.

So, their product (AB) is possible.

Let $C = [c_{ik}]_{3\times 2} = AB = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 2 & -1 & 4 \end{bmatrix}_{3\times 3} \cdot \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 5 & 1 \end{bmatrix}_{3\times 2}$

$$= \begin{bmatrix} 1\times 1 + 2\times(-1) + 0\times 5 & 1\times 2 + 2\times 4 + 0\times 1 \\ 0\times 1 + 1\times(-1) + 3\times 5 & 0\times 2 + 1\times 4 + 3\times 1 \\ 2\times 1 + (-1)\times(-1) + 4\times 5 & 2\times 2 + (-1)\times 4 + 4\times 1 \end{bmatrix}_{3\times 2}$$

$$= \begin{bmatrix} 1-2+0 & 2+8+0 \\ 0-1+15 & 0+4+3 \\ 2+1+20 & 4-4+4 \end{bmatrix}_{3\times 2} = \begin{bmatrix} -1 & 10 \\ 14 & 7 \\ 23 & 4 \end{bmatrix}_{3\times 2}$$

Multiplication of a matrix by a Scalar

- Let $A = [a_{ij}]_{m\times n}$ be any matrix and K be any constant(scalar), then the product of constant and matrix is denoted by (KA) and defined as $KA = K[a_{ij}]_{m\times n} = [K\, a_{ij}]_{m\times n}$
- **Example:**

  If K=5 and $A = \begin{bmatrix} 1 & 4 & 7 \\ 0 & 8 & 9 \\ 2 & 3 & 1 \end{bmatrix}$ then the multiplication of a matrix A and a scalar K is equal to

$$KA = 5 \cdot \begin{bmatrix} 1 & 4 & 7 \\ 0 & 8 & 9 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 5\times 1 & 5\times 4 & 5\times 7 \\ 5\times 0 & 5\times 8 & 5\times 9 \\ 5\times 2 & 5\times 3 & 5\times 1 \end{bmatrix} = \begin{bmatrix} 5 & 20 & 35 \\ 0 & 40 & 45 \\ 10 & 15 & 5 \end{bmatrix}$$

Transpose of a matrix

- Let A be a matrix of order (m×n) then the (n×m) matrix obtained from the matrix A by interchanging rows with columns is called a transpose of a matrix.
- It is denoted by $A^{'}$ or A^T
- ∴ A^T = $(j \cdot i)^{th}$ element of A' is the $(i \cdot j)^{th}$ element of A.
- **Example:**

  If $A = \begin{bmatrix} 1 & 2 & 7 \\ 3 & 1 & 4 \end{bmatrix}_{2\times 3}$ then $A^{'} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 7 & 4 \end{bmatrix}_{3\times 2}$

Notes:

If $A^{'}$ and $B^{'}$ are the transposes of A and B respectively then

(i) $\left(A^{'}\right)^{'} = A$

(ii) $(A+B)^{'} = A^{'} + B'$, A and B being of the same size

(iii) $(KA)^{'} = KA'$

(iv) $(AB)^{'} = B'A'$

Determinant

- Let $A = [a_{ij}]$ be any square matrix, the values of determinant are equal to the product of the elements along the principal diagonal minus the product of the non-principal diagonal elements.
- It is denoted by determinant A or $|A|$.
- Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then determinant value of matrix A given as

  $|A| = \det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = (ad - bc)$
- If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is of order 3 then

  $$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$
  $$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}\left(a_{21}a_{33} - a_{23a_{31}}\right) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$
- **Example:**

  If $A = \begin{bmatrix} 2 & 5 \\ 7 & -1 \end{bmatrix}$ is of order 2 then $|A| = \begin{vmatrix} 2 & 5 \\ 7 & -1 \end{vmatrix} = (2)(-1) - (5)(7) =- 2 - 35 =- 37$

Properties of Determinants

1. The value of a determinant does not change when rows and columns are interchanged.
2. If any two rows (or columns) of a determinants are interchanged, then the value of the determinant is multiply by -1.
3. If each element of any rows (or columns) has α as a common factor, then we can bring α outside the determinant.

   i.e. $\begin{vmatrix} \alpha a & b \\ \alpha c & d \end{vmatrix} = \alpha \begin{vmatrix} a & b \\ c & d \end{vmatrix}$
4. If two rows (or columns) of a determinant are identical, then the value of determinant is zero.
5. If each element in any row (or column) consists of the sum of two terms, then the determinant can be expressed as the sum of two determinants of the same order.

   i.e. $\begin{vmatrix} \alpha + a & b \\ \beta + c & d \end{vmatrix} = \begin{vmatrix} \alpha & b \\ \beta & d \end{vmatrix} + \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Minors and Co-factors

- The determinant obtained by deleting the row and the column, which interest at a particular element is called the minor.
- It is generally denoted by M_{ij}
- Consider the determinant of matrix A, as $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

  ∴ The minor of the element a_{11} i.e. $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

  The minor of the element a_{12} i.e. $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

  The minor of the element a_{22} i.e. $M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$

  And so on.

- The product of minor M_{ij} and $(-1)^{i+j}$ is the cofactor of the elements a_{ij}.
- It is denoted by c_{ij}.
- Mathematically $C_{ij} = (-1)^{i+j} M_{ij}$
- **Example:**

  $$C_{11} = Cofactor\ of\ the\ element\ a_{11} = (-1)^{1+1} M_{11} = (-1)^2 \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$
  $$= (+1)(a_{22}a_{33} - a_{23}a_{32})$$

  $$C_{12} = Cofactor\ of\ the\ element\ a_{12} = (-1)^{1+2} M_{12} = (-1)^3 \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$
  $$= -(a_{21}a_{33} - a_{23}a_{31})$$

Adjoint of a Square matrix

- Let $A = [a_{ij}]_{n\times n}$ be any square matrix, then the transpose of matrix B (i.e. B') where $B = [c_{ij}]_{n\times n} = matrix\ of\ cofactors\ of\ matrix\ A$, is called the adjoint of the matrix A and it is denoted by (Adj. A)
- If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{nn} \end{bmatrix}$

  Then $B = Matrix\ of\ Cofactors = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdots & \cdots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \cdots & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{n1} & c_{n2} & c_{n3} & \cdots & \cdots & c_{nn} \end{bmatrix}$

  $\Rightarrow B' = Transpose\ of\ the\ matrix\ B$

  $\Rightarrow adj.(A) = B' = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdots & \cdots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \cdots & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{n1} & c_{n2} & c_{n3} & \cdots & \cdots & c_{nn} \end{bmatrix}$

Singular and Non-singular Matrix

A square matrix A is called a singular matrix, if $|A|$ or det (A) is equal to zero.

Otherwise, it is a non-singular (i.e.$|A| \neq 0$)

Inverse of a square matrix

Let $A = [a_{ij}]_{n\times n}$ be a square matrix, if there exits an n-rowed square matrix B such that AB=BA=I_n, where I_n is the identity matrix of order n, then the matrix A is called to be invertible and the matrix B is called the inverse of matrix A.

Notes

1. Only square matrices have their inverses.
2. Singular matrices do not have inverse.
3. The inverse of a product is the product of the inverse taken in the reverse order. i.e. If A and B are two n-rowed non-singular matrices, then $(AB)^{-1} = B^{-1}A^{-1}$
4. If A is non-singular matrix, then the inverse of A is denoted by A^{-1} and defined as $A^{-1} = \frac{1}{|A|} adj. A$
5. If A be an n×n non-singular matrix, then $[A']^{-1} = [A^{-1}]'$
6. If A be a n-rowed non-singular matrix, then $A \cdot A^{-1} = I$

Solution of system of linear of equations using Matrix Method

- Consider a system of n linear equations in n unknowns x_1, x_2, x_3, …………, x_n
- That means,

  $a_{11}x_1 + a_{12}x_2 + \text{………..} + a_{1n}x_n = b_1$

  $a_{21}x_1 + a_{22}x_2 + \text{………..} + a_{2n}x_n = b_2$

  ……………………………………………

  ……………………………………………

  $a_{n1}x_1 + a_{n2}x_2 + \text{………..} + a_{nn}x_n = b_n$
- Rewrite the above equations in matrix form, $AX = B$………(1)

  $$\text{Where, } A = \begin{bmatrix} a_{11} & a_{12} & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & \dots & a_{nn} \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ \cdots \\ b_n \end{bmatrix} \text{ and } X = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ \cdots \\ x_n \end{bmatrix}$$
- If A is non-singular matrix. i.e. $|A| \neq 0$

  Then A^{-1} exists.

  Therefore premultiplying equation (1) by A^{-1}, we get

  $A^{-1}(AX) = A^{-1}B$

  $(A^{-1}A)X = A^{-1}B$

  $IX = A^{-1}B$

  $X = A^{-1}B$

  This gives the solution of the given linear equations.

Cramer's Rule

- Let the n linear simultaneous equations in n unknowns $x_1, x_2, \ldots\ldots\ldots, x_n$ be
  $a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots + a_{1n}x_n = b_1$
  $a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots + a_{2n}x_n = b_2$
  $a_{31}x_1 + a_{32}x_2 + \ldots\ldots\ldots + a_{3n}x_n = b_3$
  ………………………………………………
  ………………………………………………
  $a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots\ldots + a_{nn}x_n = b_n$
- Let $\Delta = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{bmatrix} \neq 0$
- Also Δ_i is the determinant obtained by replacing the i^{th} column in Δ by the elements $b_1, b_2, \ldots\ldots\ldots\ldots, b_n$.
- Thus, we have $x_i = \frac{\Delta_i}{\Delta}, i = 1,2,3, \ldots\ldots., n,\ where\ \Delta \neq 0$

Characteristic equation and roots of a square matrix

- Consider $A = [a_{ij}]_{n\times n}$ is any n-rowed square matrix and $\boldsymbol{\lambda}$ an indeterminate. The matrix A-$\boldsymbol{\lambda}$I is called the characteristic matrix of A, where I is the unit matrix of order n. The determinant $|A - \lambda I|$ is called the characteristics polynomial of A.
- Also, the equation $|A - \lambda I| = 0$ is called the characteristic equation of A and the roots of this equation are called the characteristic roots or values or Eigen values of the matrix A.
- If $\boldsymbol{\lambda}$ is an eigen value of the matrix A, then $|A - \lambda I| = 0$ and the matrix $(A - \lambda I)$ is singular.
- Therefore, there exists a non-zero vector A such that $(A - \lambda I)X = 0\ \ OR\ \ AX = \boldsymbol{\lambda} X$
- Characteristic Vector: If $\boldsymbol{\lambda}$ is an eigen value of a square matrix A, then a non zero vector X such that AX=$\boldsymbol{\lambda}$X is called an eigen vector of a corresponding to the eigen value $\boldsymbol{\lambda}$.

Cayley-Hamilton Theorem

Statement:

Every square matrix satisfies its own characteristic equation.

OR

If for a square matrix A or order n, $|A-\lambda I| = (-1)^n[\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots\ldots\ldots + a_n]$ then the matrix equation $X^n + a_1X^{n-1} + a_2X^{n-2} + \ldots\ldots\ldots + a_nI = 0$ is satisfied by X=A

$\therefore\ A^n + a_1A^{n-1} + a_2A^{n-2} + \ldots\ldots\ldots + a_nI = 0$

Proof:

- The all elements of $(A-\lambda I)$ are at most of the first degree in $\boldsymbol{\lambda}$, the elements of $adj.(A-\lambda I)$ are at most degree (n-1) in $\boldsymbol{\lambda}$.
- Therefore $adj.(A-\lambda I)$ can be written as a matrix polynomial in $\boldsymbol{\lambda}$ such that

$$adj.(A-\lambda I) = B_0\lambda^{n-1} + B_1\lambda^{n-2} + B_2\lambda^{n-3} + \ldots\ldots\ldots + B_{n-1}$$

- Where $B_0, B_1, B_2, \ldots\ldots\ldots, B_{n-1}$ are square matrix of order n, whose elements are functions of $a_{ij}^{'}s$.

$$\therefore\ (A-\lambda I)adj.(A-\lambda I) = |A-\lambda I|\cdot I$$

$$\Rightarrow (A-\lambda I)[B_0\lambda^{n-1} + B_1\lambda^{n-2} + B_2\lambda^{n-3} + \ldots\ldots\ldots + B_{n-1}]$$

$$= (-1)^n[\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots\ldots\ldots + a_n]I$$

- Comparing the co-efficient of like powers of $\boldsymbol{\lambda}$ on both sides, we get

$$-IB_0 = (-1)^nI$$

$$AB_0 - IB_1 = (-1)^na_1I$$

$$AB_1 - IB_2 = (-1)^na_2I$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$AB_{n-1} = (-1)^na_nI$$

- On multiplying the equations by $A^n, A^{n-1}, \ldots\ldots\ldots\ldots, I$ respectively and adding, we get

$$0 = (-1)^n[A^n + a_1A^{n-1} + a_2A^{n-2} + \ldots\ldots\ldots + a_nI]$$

- Hence,

$$A^n + a_1A^{n-1} + a_2A^{n-2} + \ldots\ldots\ldots + a_nI = 0$$

Application of Matrices in solving Pharmacokinetic equations

- In matrix designed drug delivery system, the drug is homogeneously dispersed either at the molecular scale or solid particle within a polymeric medium. In comparison to reservoir systems, the manufacture of matrix designed drug delivery system is more straight forward and may be performed using a number of different approaches.
  1) The rate of release of therapeutic agents from reservoir transdermal system may be conveniently described by the following equation,

     $$\frac{dM}{dt} = \frac{K_{m/r}K_{a/m}D_aD_m}{K_{m/r}D_mh_a + K_{a/m}D_ah_m} \times C_R$$

     Where C_R = The drug concentration present in the matrix, $K_{m/r}$ and $K_{a/m}$ = The reservoir or adhesive partition coefficient, D_m and D_a = The diffusion coefficients of the drug in the rate controlling membrane and the adhesive layer, h_m and h_a = The thickness of the rate controlling membrane and the adhesive layer.

  2) **Drug dissolved within the polymer matrix:**
     In case of drug, it is dissolved in the polymer matrix. That means, hexetidine dissolved in polyvinyl chloride, the release rate of drug $\left(\frac{dM}{dt}\right)$ from a slab geometry may be described using the following equation, over the first 60% of the release rate.

     $$\frac{dM}{dt} = 2M_0\left(\frac{D}{\pi L^2 t}\right)^{0.5}$$

     Where, M_0 = The total amount of drug dissolved in the polymer matrix

     M = The mass of drug release at time t

     L = The thickness of the Mab

     D = The diffusion coefficient

  3) Rust is formed when there is a chemical reaction between iron and oxygen. The compound that is formed is redish-brown that cover the iron object. Rust is an iron whose chemical formula is Fe_2O_3, so the chemical equation for rust is $Fe + O_2 \rightarrow Fe_2O_3$
     Balance the equation.

Solution:

Using unknowns $x, y\ and\ z$, to balance the given equation in this manner

$xFe + yO_2 \rightarrow zFe_2O_3$……………………. (1)

Next, we compare the number of Fe and O atoms of the reactants with the number of the products. We obtain two linear equations,

$Fe: x = 2z$

$O: 2y = 3z$

Rewriting above equations, we get a homogeneous linear system in three unknowns

$i.e.$

$x - 2z = 0$ & $2y - 3z = 0$

Again, we have the Argumanted matrix

$$(A/B) = \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

After solving, we get $x = \frac{4}{3}y, z = \frac{x}{2}$ and then $x = 4, y = 3$ & $z = 2$

Then equation is $4Fe + 3O_2 \rightarrow 2Fe_2O_3$