Shree H. N. Shukla College of Science

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F.Y.B.SC. SEM-I (CBCS)

SUBJECT: Mathematics

PAPER: 101

Unit: 2

Indeterminate Forms & Differential Equation of first order & first degree

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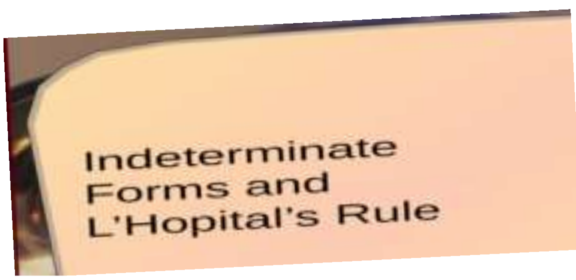
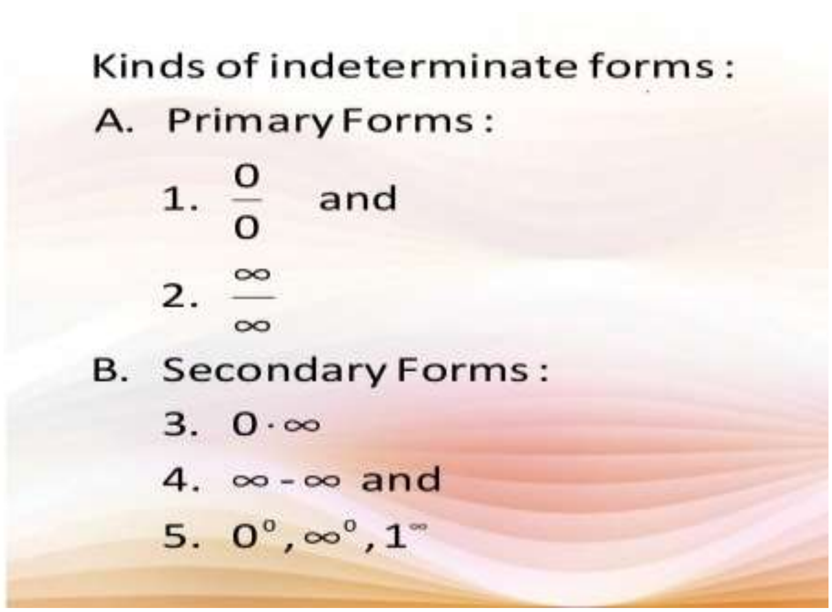
INTRODUCTION

☑ Indeterminate forms

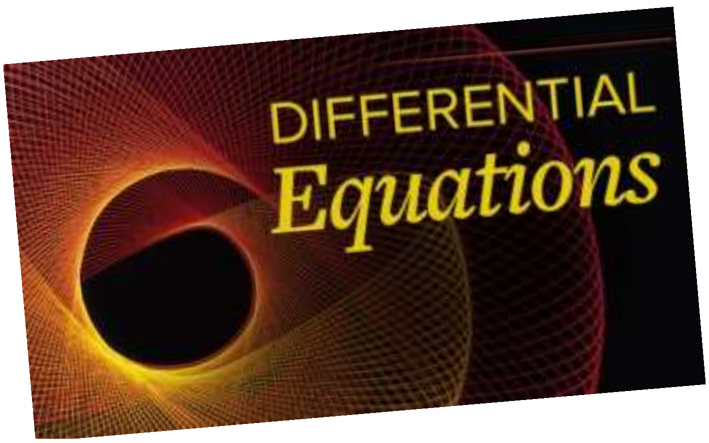
- L'hospitals rules for various indeterminate forms(without proof)
- Various indeterminate forms like $\frac{0}{0}$ form, $\frac{\infty}{\infty}$ form, $0 \cdot \infty$ form, $\infty - \infty$ form, 0^0 form, ∞^0 form

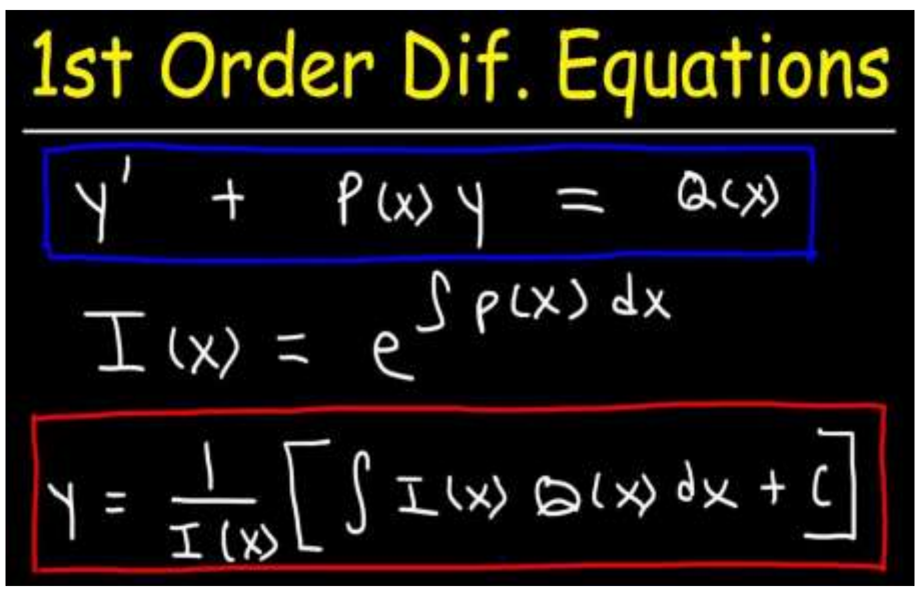




☑ Differential equations of first order and first degree

- Definitions and method of solving of Differential equation of the form variable separable
- Homogeneous differential equation
- Linear differential equations of first order and first degree





TITLE EXPLANATION

- In this unit we are going to learn and discuss about different types of indeterminate forms & Differential equations.

- ## Indeterminate forms;

  In calculus and other branches of mathematical analysis, limits involving an algebraic combination of functions in an independent variable may often be evaluated by replacing these functions by their limits; if the expression obtained after this substitution does not provide sufficient information to determine the original limit, then the expression is called an indeterminate form.

- More specifically, an indeterminate form is a mathematical expression involving 0, 1 and ∞.

- ## Differential equation;

  In mathematics, a **differential equation** is an equation that relates one or more functions and their derivatives.

- An equation involving a dependent variable and its derivatives with respect to one or more independent variables is called a Differential Equation.

- ## First order and first degree differential equation;

  A differential equation of first order and first degree can be written as $f(x, y, dy/dx) = 0$.

LEARNING OUTCOME

Indeterminate forms:

- → Recognize when to apply L'Hospitals rule.
- → Identify indeterminate forms produced by quotients, products, subtractions, and powers, and apply L'Hospitals rule in each case.
- → Describe the relative growth rates of functions.

Differential equation:

- → Recognize differential equations that can be solved by each of the three methods – direct integration, separation of variables and integrating factor method – and use the appropriate method to solve them
- → Use an initial condition to find a particular solution of a differential equation, given a general solution
- → Check a solution of a differential equation in explicit or implicit form, by substituting it into the differential equation
- → Understand the terms 'exponential growth/decay', 'proportionate growth rate' and 'doubling/halving time' when applied to population models, and the terms 'exponential decay', 'decay constant' and 'half-life' when applied to radioactivity
- → Solve problems involving exponential growth and decay.

Indeterminate forms:

- If $\emptyset(x) = \frac{f(x)}{g(x)}$ is function of x and f(x) → 0, g(x) → 0 as x → a then $\emptyset(a) = \frac{0}{0}$ represent the indeterminate form $\frac{0}{0}$.
- Similarly f(x) → ∞, g(x) → ∞ as x → a then $\emptyset(a) = \frac{\infty}{\infty}$, also represent the indeterminate form $\frac{\infty}{\infty}$.
- To evaluate these indeterminate forms means to find $\lim_{x \to a} \frac{f(x)}{g(x)}$, provided it exists.
- Other indeterminate forms are $0 \cdot \infty, \infty - \infty, 0^{\infty}, 1^{\infty}, 0^{0}, \infty^{\infty}$, etc.
- For example,

$\lim_{x\to 0} \frac{sinx}{x}, \lim_{x\to 0} \frac{logsinx}{cotx}, \lim_{x\to 0}(cosecx - cotx),$
$\lim_{x\to 0} x^{x}, \lim_{x\to 0} sinxlog\frac{1}{x}$
Are $\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0^{0}, 0 \cdot \infty$ forms respectively.

- $\frac{0}{0}$ *and* $\frac{\infty}{\infty}$ are fundamental indeterminate forms, all other indeterminate forms can be converted into $\frac{0}{0}$ *and* $\frac{\infty}{\infty}$ form.
- To evaluate these indeterminate forms, French Mathematician L'Hospital, find a method which is known as L'Hospitals rule.

SR. NO.	QUESTION	ANSWER
1	If $\emptyset(x) = \frac{f(x)}{g(x)}$ is function of x and f(x) → 0, g(x) → 0 as x → a then $\emptyset(a) = \frac{0}{0}$ represent the indeterminate form ______	$\frac{0}{0}$
2	Write down the fundamental indeterminate forms.	$\frac{0}{0}$ *and* $\frac{\infty}{\infty}$
3	1^{∞} is an indeterminate form.(T/F)	**True**
4	Who evaluated indeterminate forms?	**French Mathematician L'Hospital**

THEOREM-1:

L'Hospitals rule for $\frac{0}{0}$ form. (Without proof)

Statement:

If f and g are two real functions defined on [α, β] and for a ϵ (α, β).

(i) f and g are continuous in [α, β]

(ii) f(a)=0=g(a)

(iii) $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ exist then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$

THEOREM-2:

L'Hospitals rule for $\frac{\infty}{\infty}$ form (Without proof).

Statement:

If f and g are two real functions defined on [α, β] and for a ϵ (α, β).

(i) f and g are continuous in [α, β]

(ii) f and g are derivable in (α, β), for x∈(α, β)-{a} and g'(x)≠0

(iii) f(x)→∞, g(x)→∞ as x→a

(iv) $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ exist then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$

NOTE:

The L'Hospitals rule for $\frac{\infty}{\infty}$ form can be extended to higher derivatives as done in the case of $\frac{0}{0}$ form.

SR. NO.	QUESTION	ANSWER
1	French Mathematician L'Hospital, find a method which is known as……	**L'Hospitals rule**
2	The L'Hospitals rule for $\frac{\infty}{\infty}$ form can be extended to___________ as done in the case of $\frac{0}{0}$ form	**higher derivatives**

EXAMPLE-1:

Find

$$\lim_{x\to 0}\frac{x\cos x-\log(1+x)}{x^2}$$

SOLUTION:

$$\lim_{x\to 0}\frac{x\cos x-\log(1+x)}{x^2}; \qquad \frac{0}{0}\ form$$

$$=\lim_{x\to 0}\frac{\cos x - x\sin x-\frac{1}{1+x}}{2x}; \qquad \frac{0}{0}\ form$$

$$= \lim_{x\to 0} \frac{-sinx - [sinx + xcosx] + \frac{1}{(1+x)^2}}{2}$$

$$= \frac{1}{2}$$

EXAMPLE-2:

Find

$$\lim_{x\to a} \frac{\log(x-a)}{a\log(e^x - e^a)}$$

SOLUTION:

$$\lim_{x\to a} \frac{\log(x-a)}{alog(e^x - e^a)}; \qquad \frac{\infty}{\infty}\ form$$

$$= \lim_{x\to a} \frac{1/x-a}{e^x/(e^x - e^a)}; \qquad \frac{\infty}{\infty}\ form$$

$$= \lim_{x\to a} \frac{e^x - e^a}{e^x \cdot (x-a)}; \qquad \frac{0}{0}\ form$$

$$= \lim_{x\to a} \frac{e^x}{e^x(x-a) + e^x}$$

$$= 1$$

EXAMPLE-3:

Evaluate

$$\lim_{x\to 0} \frac{logx^2}{cotx^2}$$

SOLUTION:

It is a $\frac{\infty}{\infty}$ form and therefore

$$\lim_{x\to 0}\frac{logx^2}{cotx^2} = \lim_{x\to 0}\frac{{}^{1}/_{x^2}\cdot 2x}{-cosec^2x^2 2x}$$

$$= \lim_{x\to 0}\frac{-1}{x^2cosec^2x^2}$$

$$= \lim_{x\to 0}\frac{-sinx^2}{x^2}; \qquad \frac{0}{0}\ form$$

$$= \lim_{x\to 0}\frac{cosx^2\cdot 2x}{2x}$$

$$= \lim_{x\to 0} cosx^2$$

$$= -1$$

EXAMPLE-4:

Evaluate

$$\lim_{x\to\frac{\pi}{2}}(1-\sin x)\tan x$$

SOLUTION:

This is $0\cdot\infty$ form, therefore we have,

$$\lim_{x\to\frac{\pi}{2}}(1-\sin x)\tan x = \lim_{x\to\frac{\pi}{2}}\frac{1-\sin x}{\cot x}; \qquad \frac{0}{0}\ form$$

$$= \lim_{x\to\frac{\pi}{2}}\frac{-\cos x}{-cosec^2x}$$

$$= \lim_{x \to \frac{\pi}{2}} \cos x \cdot sin^2 x$$

$$= 0$$

EXAMPLE-5:

Evaluate

$$\lim_{x \to 0} \left(cot^2 x - \frac{1}{x^2} \right)$$

SOLUTION:

This is $\infty - \infty$ form, therefore we have,

$$\lim_{x \to 0} \left(cot^2 x - \frac{1}{x^2} \right) = \lim_{x \to 0} \frac{x^2 cos^2 x - sin^2 x}{x^2 sin^2 x}$$

$$= \lim_{x \to 0} \frac{x^2 cos^2 x - sin^2 x}{x^4} \cdot \frac{x^2}{sin^2 x}$$

$$= \lim_{x \to 0} \frac{x^2 cos^2 x - sin^2 x}{x^4} \qquad \left(\because \lim_{x \to 0} \frac{x^2}{sin^2 x} = 1 \right)$$

$$= \lim_{x \to 0} \frac{2x cos^2 x - 2x^2 \cos x \sin x - 2 \sin x \cos x}{4x^3}$$

$$= \lim_{x \to 0} \frac{2x cos^2 x - x^2 \sin 2x - \sin 2x}{4x^3}; \qquad \frac{0}{0}\ form$$

$$= \lim_{x \to 0} \frac{2cos^2 x - 4x \sin x \cos x - 2x \sin 2x - 2x^2 \cos 2x - 2 \cos 2x}{12x^2}$$

$$= \lim_{x \to 0} \frac{2cos^2 x - 2x \sin 2x - 2x \sin 2x - 2x^2 \cos 2x - 2 \cos 2x}{12x^2}$$

$$= \lim_{x\to 0} \frac{2cos^2 x - 4x\sin 2x - 2x^2\cos 2x - 2\cos 2x}{12x^2}; \qquad \frac{0}{0}\ form$$

$$= \lim_{x\to 0} \frac{-4\cos x\sin x - 4\sin 2x - 8x\cos 2x - 4x\cos 2x + 4x^2\sin 2x + 4\sin 2x}{24x}$$

$$= \lim_{x\to 0} \frac{-2\sin 2x - 12x\cos 2x + 4x^2\sin 2x}{24x}; \qquad \frac{0}{0}\ form$$

$$= \lim_{x\to 0} \frac{-4\cos 2x - 12\cos 2x + 24x\sin 2x + 8x\sin 2x + 8x^2\cos 2x}{24}$$

$$= \frac{-4-12}{24} = \frac{-16}{24} = \frac{-2}{3}$$

EXAMPLE-6:

Evaluate

$$\lim_{x\to\frac{\pi}{2}} (\sin x)^{\tan x}$$

SOLUTION:

This is 1^{∞} form, therefore we have,

$$y = (\sin x)^{\tan x} \Rightarrow \log y = \tan x \cdot \log\sin x = \frac{\log\sin x}{\cot x}$$

$$\therefore \lim_{x\to\frac{\pi}{2}} \log y = \lim_{x\to\frac{\pi}{2}} \frac{\log\sin x}{\cot x}; \qquad \frac{0}{0}\ form$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\cot x}{-coses^2 x}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\cos x}{\sin x} \cdot (-sin^2 x)$$

$$= \lim_{x \to \frac{\pi}{2}} -\sin x \cdot \cos x = 0$$

$$\therefore \lim_{x \to \frac{\pi}{2}} \log y = 0$$

$$\therefore \lim_{x \to \frac{\pi}{2}} y = e^0 = 1$$

EXAMPLE-7:

Evaluate

$$\lim_{x \to 0} (\cot x)^{\frac{1}{\log x}}$$

SOLUTION:

This is ∞^0 form therefore,

Let $y = (\cot x)^{\frac{1}{\log x}} \Rightarrow \log y = \frac{1}{\log x} \cdot \log \cot x$

$$\therefore \lim_{x \to 0} \log y = \lim_{x \to 0} \frac{\log \cot x}{\log x}; \qquad \frac{\infty}{\infty} \ form$$

$$= \lim_{x \to 0} \frac{\frac{-cosec^2 x}{\cot x}}{1/x}$$

$$= \lim_{x \to 0} \frac{-x}{\sin x} \cdot \frac{1}{\cos x}$$

$$= -\frac{1}{\cos 0} \qquad \left(\because \lim_{x \to 0} \frac{x}{\sin x} = 1\right)$$

$$\therefore \lim_{x \to 0} \log y = -1$$

$$\therefore \lim_{x\to 0} y = e^{-1} = \frac{1}{e}$$

EXAMPLE-8:

If

$$\lim_{x\to\frac{\pi}{2}} \frac{cos^2x}{a - bcosecx} = 1$$

Then find value of a and b.

SOLUTION:

Here given that $\lim_{x\to\frac{\pi}{2}} \frac{cos^2x}{a-bcosecx}$ exists and since $\lim_{x\to\frac{\pi}{2}} cos^2x = 0$, by L'Hospital's rule denominator $a - bcosec\frac{\pi}{2} \to 0\ as\ x \to \frac{\pi}{2}$.

Hence $a - bcosec\frac{\pi}{2} = 0 \Rightarrow a - b = 0 \Rightarrow a = b \qquad \ldots\ldots\ldots\ldots..(i)$

Now by L'Hospital rule,

$$\lim_{x\to\frac{\pi}{2}} \frac{cos^2x}{a - bcosecx} = \lim_{x\to\frac{\pi}{2}} \frac{-2\sin x \cos x}{b\cot x cosecx} = \lim_{x\to\frac{\pi}{2}} \frac{-2}{b} sin^3x = \frac{-2}{b}$$

But given that

$$\lim_{x\to\frac{\pi}{2}} \frac{cos^2x}{a - bcosecx} = 1$$

$$\therefore \frac{-2}{b} = 1$$

$$\therefore b = -2$$

From result (i), we get

$$a = b = -2$$

REMARKS:

We can deduce from L'Hospital's rule that,

1) If $f'(a)$ and $g'(a)$ are defined and $\lim_{x\to a} f'(x)$ and $\lim_{x\to a} g'(x)$ exists then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{f'(a)}{g'(a)}$$

2) If $f'(a)=g'(a)=0$ and second derivative exists for f and g then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=\lim_{x\to a}\frac{f''(x)}{g''(x)} \qquad (g''(x)\neq 0)$$

Similarly,

$$f^{n-1}(a)=g^{n-1}(a)=0 \; and \; g''(x)\neq 0 \; then$$

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f^{n}(x)}{g^{n}(x)}$$

EXERCISE-A

Evaluate the following:

1) $\lim_{x\to 0}\frac{\sin x - x\cos x}{\sin x - x}$
2) $\lim_{x\to 1}\frac{\log x}{x-1}$
3) $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^{x}-1}\right)$
4) $\lim_{x\to\frac{\pi}{2}}(\sec x-\tan x)$

Definition: Differential equation

- Differential equation is an equation which involves differential coefficients OR differentials.

  Thus,

1) $e^x\,dx = e^y\,dy$
2) $\frac{d^2x}{dt^2} + n^2x = 0$
3) $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} \Big/ \frac{d^2y}{dx^2} = C$

Are all examples of Differential equations.

Order of Differential Equation:

Order of Differential Equation is the order of the highest derivative appearing in it.

Degree of a Differential Equation:

Degree of a Differential Equation is the degree of the highest derivative occurring in it, after the equation has been expressed in a form free from radicals and Factions as far the derivatives are concerned.

Thus, from the examples above,
(1) is of the first order and first degree;
(2) is of the second order and first degree;
(3) is of the second order and second degree.

Formation of a differential equation:
An ordinary differential equation is formed in an attempt to arbitrary constant from a relation in the variables and constant. Example-9: Form the differential equation of simple harmonic A

eliminate certain

motion given by