Shree H. N. Shukla college of science, Rajkot
Riemann Integration
Unit 1(a)

➢ Title Justification Of Riemann Integration

- Let f : [a, b] → R be a bounded (not necessarily continuous) function on a compact (closed, bounded) interval. We will define what it means for f to be Riemann integrable on [a, b] and, in that case, define its Riemann integral R b a f.
- The integral of f on [a, b] is a real number whose geometrical interpretation is the signed area under the graph y = f(x) for a ≤ x ≤ b.
- This number is also called the definite integral of f. By integrating f over an interval [a, x] with varying right end-point, we get a function of x, called the indefinite integral of f.
- The most important result about integration is the fundamental theorem of calculus, which states that integration and differentiation are inverse operations in an appropriately understood sense.
- Among other things, this connection enables us to compute many integrals explicitly. Inerrability is a less restrictive condition on a function than differentiability.
- Roughly speaking, integration makes functions smoother, while differentiation makes functions rougher.
- For example, the indefinite integral of every continuous function exists and is differentiable, whereas the derivative of a continuous function need not exist (and generally doesn't).
- The Riemann integral is the simplest integral to define, and it allows one to integrate every continuous function as well as some not-too-badly discontinuous functions.
- There are, however, many other types of integrals, the most important of which is the Lebesgue integral.

- The Lebesgue integral allows one to integrate unbounded or highly discontinuous functions whose Riemann integrals do not exist, and it has better mathematical properties than the Riemann integral.
- The definition of the Lebesgue integral requires the use of measure theory, which we will not describe here. In any event, the Riemann integral is adequate for many purposes, and even if one needs the Lebesgue integral, it's better to understand the Riemann integral first.
- **Theme OR Points Of The Chapter**
  - Partitions and Riemann sums,
  - Upper and lower R-integrals, R-integrability,
  - The integral as limit, Some classes of integrable functions ,
  - Properties of integrable
  - Function, Properties of R-integrable function.,
  - Statement of **Darboux's theorem** (without proof)

Shree H. N. Shukla college of science, Rajkot
Riemann Integration
Unit 1(a)

1) Define:- Partition of a set:-

Set I = [a, b] be a closed interval. Then by a partition of I are mean a finite set $P = \{ x_0, x_1, \ldots x_n\}$ of real numbers having the property that

$a = x_0 < x_1 < \ldots x_{n-1} < x_n = b$

Example. P= {1, 3, 5, 7, 9} is a partition of [1, 9]

Note:-

(1) If $P = \{x_0, x_1, \ldots x_n\}$ is a partition of [a,b] then $[x_0, x_1], [x_1, x_2] \ldots [x_{n-1}, x_n]$ are called **subintervals.**

(2) $[x_{i-1}], x_i]$ is called **i^{th} subinterval** of [a, b].

(3) $x_i - x_{i-1}$ is called **length of i^{th} subintervals it is denoted by δ_i.**

(4) Thus $\delta_i = x_i - x_{i-1}$

(5) $\sum_{i=1}^{n} \delta i$ =b – a = length of [a, b]

(6) max {δi/ I = 1, 2 ….n } is called norm of partition P and is denoted by ||P||.

(7) if $n \to \infty$ then $||P|| \to 0$

2) Define: - Refinement of partition.

Set P and P* are two partition of [a,b] and $P \subset P^*$ then P* is called a refinement of P or P* refines P or P* is refine then P or P* is interior of P.

Note:-

(1) $||P^*|| \le ||P||$

(2) $P \cup P^*$ is also a partition and it is a refinement of P and P*

3) Define: Low Riemann sum and upper Riemann sum

Set f be a bounded function define on [a, b] and let $P = \{x_0, x_1, \ldots x_n\}$ is a partition of [a, b].Let mi and Mi are glb and lub of function f in i^{th} interval respectively

Then the sum $\sum_{i=1}^{n}$ miδi and $\sum_{i=1}^{n}$ Miδi

Can be defined and these sums are called **lower Riemann sum and upper Riemann sum** respectively.

They are denoted by **L (P, f) and U (P, f**), respectively.

Thus, L(P, f) = $\sum_{i=1}^{n}$ miδi and U(P, f) = $\sum_{i=1}^{n}$ Miδi

Note:-

(1) If m and M are glb and lub of [a, b] and mi and Mi are glb and lub of f in the i^{th} interval then m ≤ Mi ≤ M

(2) If f is increasing function in [a, b] then M = f(a) and M= f(b) and if f is decreasing function in [a, b] then M = f(b) and m = f(a)

Ex: 1 if f(x) = $\frac{20}{x}$ where x ∈ [2, 20] then find L (P, f) and U (P, f) by taking partition P = {2, 4, 5, 20}.

Ex.:2 If f(x) = x, where x ∈ [0, 1] then find L (P, f) and U (P, f) by taking partition P = $\left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$.

Ex.:3 if f(x) = x, where x ∈ [0, 1] then find L (P, f) and U (P, f) by taking partition P = $\left\{0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\right\}$.

Solution:-

Here f(x) = x ; x ∈ [2, 20] is a decreasing function

For i^{th} subinterval $[x_{i-1}, x_i]$

mi = f(xi) ; Mi = f(x_i-1)

Here P = $\left\{0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\right\}$

by the I = [0, 1]

the I partition's are

$I_1 = \left[0, \frac{1}{5}\right]$, $I_2\left[\frac{1}{5}, \frac{2}{5}\right]$, $I_3 = \left[\frac{2}{5}, \frac{3}{5}\right]$, $I_4 = \left[\frac{3}{5}, \frac{4}{5}\right]$, $I_5 = \left[\frac{4}{5}, 1\right]$

For $I_1 = \left[0, \frac{1}{5}\right]$

$m_1 = \frac{1}{5}$, $M_1 = 0$, $\delta_1 = \frac{1}{5}$

For $I_2 = \left[\frac{1}{5}, \frac{2}{5}\right]$

$m_2 = \frac{2}{5}$, $M_2 = \frac{1}{5}$, $\delta_2 = \frac{1}{5}$

For $I_3 = \left[\frac{2}{5}, \frac{3}{5}\right]$

$m_3 = \frac{3}{5}$, $M_3 = \frac{2}{5}$, $\delta_3 = \frac{1}{5}$

For $I_4 = \left[\frac{3}{5}, \frac{4}{5}\right]$

$m_4 = \frac{4}{5}$, $M_4 = \frac{3}{5}$, $\delta_4 = \frac{1}{5}$

For $I_5 = \left[\frac{4}{5}, 1\right]$

$m_5 = 1$, $M_5 = \frac{4}{5}$, $\delta_5 = \frac{1}{5}$

Now, **L (P, f) =** $\sum_{i=1}^{5}$ **$m_i\delta_i$**

$= m_1\delta_1 + m_2\delta_2 + m_3\delta_3 + m_4\delta_4 + m_5\delta_5$

$= \frac{1}{5}\cdot\frac{1}{5} + \frac{2}{5}\cdot\frac{1}{5} + \frac{3}{5}\cdot\frac{1}{5} + \frac{4}{5}\cdot\frac{1}{5} + 1\cdot\frac{1}{5}$

$= \frac{1}{25} + \frac{2}{25} + \frac{3}{25} + \frac{4}{25} + \frac{5}{25}$

$= \frac{15}{25}$

$= \frac{3}{5}$

Now **U (P, f) =** $\sum_{i=3}^{5}$ **$M_i\delta_i$**

$= M_1\delta_1 + M_2\delta_2 + M_3\delta_3 + M_4\delta_4 + M_5\delta_5$

$= 0\cdot\frac{1}{5} + \frac{1}{5}\cdot\frac{1}{5} + \frac{2}{5}\cdot\frac{1}{5} + \frac{3}{5}\cdot\frac{1}{5}\cdot\frac{4}{5}\cdot\frac{1}{5}$

$= \frac{1}{25} + \frac{2}{25} + \frac{3}{25} + \frac{4}{25} + \frac{5}{25}$

$= \frac{10}{25}$

$= \frac{2}{5}$

Theorem 1st: Let f be a founded function defined on [a, b]. Let m and M are glb and lub of in [a,b] respectively, then there exists a partition P of [a, b] such that m (b – a) ≤ L (P, f) ≤ M (b – a)

Proof:-

Here f be a bounded function defined on [a, b] ,

m and M are glb and lub of f in [a, b] respectively.

Let P = {a = x_0, x_1, …. x_{n-1} = b} is any partition of [a, b]

Let mi and Mi are glb and lub of f in i^{th} sub interval of {x_{i-1}, x_i} of [a,b]

and δ_i is length of {x_{i-1}, xi}

We know that

m ≤ mi ≤ Mi ≤ M ; ∀ I ≠ 1, 2 …. n

mδi ≤ miδi ≤ Miδi ≤ Mδi ; ∀ I 1, 2 …. N (∵ multiplying by δi > 0)

By adding

$$\sum_{i=1}^{n} m\delta i \le \sum_{i=1}^{n} mi\delta i \le \sum_{i=1}^{n} Mi\delta i \le \sum_{i=1}^{n} M\delta i$$

$$m\sum_{i=1}^{n} \delta_i \le \sum_{i=1}^{n} mi\delta i \sum_{i=1}^{n} Mi\delta i \le \sum_{i=1}^{n} M\delta i$$

$$m (b - a) \le \sum_{i=1}^{n} mi\delta_i \le \sum_{i=1}^{n} Mi\delta i \le M (b - a)$$

m (b – a) ≤ L(P, f) ≤ U (P, f) ≤ M (b – a)

Theorem 2nd: set f be bounded function defined on [a, b] if P and P* are two partition of [a, b] such that P⊂P*

Then L (P, f) ≤ L (P*, f) ≤ U (P*, f) ≤ U (P, f).

Proof:-

Here f be a bounded function defined on [a, b]

Let P = {a = x_0, x_1, …... x_n =b} and p* = {a = x_0, ….. x_n = b, y_1, y_2, ….y_m}

Are two partition of [a, b]

Here y_1, y_2,…..y_m are somewhere in between {x_0…….x_n}

Then clearly $P \subset P^*$

Consider $P_1 = P \cup \{y_1\}$

Then P_1 is partition of [a, b]

Set m and M are glb and lub of f in [a, b],

mi and Mi are glb and lub of $\{x_{i-1}, x_i\}$

Now, consider $x_{i-1}, \leq y_1 \leq x_i$ for some i = 1….n

Let w_1 and w_2 are glb and lub of f in $[x_{i-1}, y]$ & $[y, x_i]$.

Then $w_1 \leq mi$, $w_2 \leq Mi$

Now for partition P_1,

$$U(P, f) = \sum_{i=1}^{n} Mi\delta i + w_1 (y - x_{i-1}) + w_2 (x_i - y) + \sum_{i=1}^{n} Mi\delta i$$

$$\leq \sum_{i=1}^{n} Mi\delta i + Mi (y-xi) + Mi(y - x_{i-1})$$

$$= \sum_{i=1}^{n} Mi\delta i + Mi (xi - x_{i-1}) + \ldots + Mi (y - x_{i-1})$$

$$= \sum_{i=1}^{n} Mi\delta i + Mi\delta i + \ldots + Mi\delta i$$

$$= \sum_{i=1}^{n} Mi\delta i$$

$U(P^*, f) \leq U(P, f)$

$= U(P, f)$

Similarly by considering

$P_2 = P_1 \cup (y_2)$, $P_3 = P_2 \cup (y_3)$

$Pm = P_{m-1} \cup \{y_m\}$

By considering are prove that

$U(P_2, f) \leq U(P_1, f)$

$U(m, f) \leq U(Pm\text{-}1, f)$

$U(Pm, f) \leq U(P, f)$

$U(P^*, f) \leq U(P, f)$ -------- (A)

Similarly are can prove that

L (P, f) ≤ L (P*, f) -------- (B)

Now for P* According to theorem (1)

L (P*, f) ≤ U (P*, f) -------(C)

From (A), (B) & (C)

L (P, f) ≤ L (P*, f)

L (P, f) ≤ L (P*, f) ≤ U (P*, f) ≤ U (P, f)

4) Definition: Upper Riemann integration:

Let bounded function f is defined on [a, b].

The set of all upper Riemann sum i.e. $\left\{U\left(P_i, f\right)\right\}_{i=1}^{\infty}$ is lower bounded. The greatest lower bound of set $\left\{U\left(P_i, f\right)\right\}_{i=1}^{\infty}$ is called Riemann integral of function f on [a, b] and it is denoted by

$$\int_a^{\bar{b}} f(x)dx = \text{glb}\left\{U\left(P_i, f\right)\right\}_{i=1}^{\infty}$$

5) Definition: Lower Riemann integration:

Let bounded function f be defined on [a, b]. The set of all lower Riemann sum i.e. $\left\{L\left(P_i, f\right)\right\}_{i=1}^{\infty}$ is upper bounded. The last upper bounded of set is called lower Riemann integral of function f on [a, b] and it is denoted by

$$\int_{\underline{a}}^{b} f(x)dx = \text{lub}\left\{L\left(P_i, f\right)\right\}_{i=1}^{\infty}$$

6) Definition: Riemann integration:

Let f be bounded function defined on [a, b]. In $\underline{\int_{a}^{b}} f(x)dx$ and $\overline{\int_{a}^{b}} f(x)dx$ exist and

$\underline{\int_{a}^{b}} f(x)dx = \overline{\int_{a}^{b}} f(x)dx$ then function f is said to be **Riemann integrable** on [a, b]

and it is denoted **by f $\in$ $R_{[a,b]}$** or **R-integrable**.

$\int_{a}^{b} f(x)dx = \overline{\int_{a}^{b}} f(x)dx = \underline{\int_{a}^{b}} f(x)dx$ **(Darboux's Condition of inerrability)**

Theorem: 3 Let f be a continuous bounded function defined over [a, b].

For any two partitions P_1 and P_2 of [a, b]. Then Prove that $L(P_1, f) \le U(P_2, f)$

Proof: Let $P = P_1 \cup P_2$

$\therefore$ P is refinement of both partitions P_1 and P_2.

Then by theorem (2) we have

$L(P_1, f) \le L(P, f)$ and $U(P, f) \le U(P_2, f)$ ……. (1)

Also we know that,

$L(P, f) \le U(P, f)$ ……. (2)

From (1) and (2), we have

$\mathbf{L(P_1, f) \le L(P, f) \le U(P, f) \le U(P_2, f)}$

$\therefore L(P_1, f) \le U(P_2, f)$

Ex.:4 If f: [a, b] → R is a constant function them show that f is Riemann integrable over [a, b] or prove that every constant function on any closed interval is Riemann integrable is Riemann integrable or if the f is defined as f(x) = K ∀x ∈ [a, b] and K is constant than prove that f ε R (a, b)

Also find $\int_a^b f dx$.

Proof:-

Here f is defined as f(x) = K ∀x ∈ [a, b] and k is a constant.

Set P = {a= x_0 ….. x_n = b} is a partition of [a, b].

Let mi and Mi are glb and lub of the f in I = [x_{i-1}, x_i]

andδi is a length of the [x_{i-1}, xi]

Now, mi = glb {f(x)| $x_{i-1} \le x \le x_i$} &Mi = lub{ f(x) | $x_{i-1} \le x \le x_i$}

= glb (k} = lub {k}

= k = k

L (P, f) = $\sum_{i=1}^{n}$ miδI & U (P, f) = $\sum_{i=1}^{n}$ Miδi

= $\sum_{i=1}^{n}$ kδi = $\sum_{i=1}^{n}$ kδi

= k $\sum_{i=1}^{n}$ δi = k $\sum_{i=1}^{n}$ δi

= k (b – a) = k (b – a)

Thus $\underline{\int_a^b} f dx = \overline{\int_a^b} f dx$

f∈ R [a, b] and

$\int_a^b f dx = \underline{\int_a^b} f dl = \overline{\int_a^b} f dx$ = **k (b – a)**

Ex.:5 A function f is defined as f(x) = 0; if x is a rational number

=1; if x is a irrational number

Ex.:6 A function f is defined as f(x) = -1; if x is a rational number

=1; if x is a irrational number

Solution:-

Set P = {a = x_0 …. x_n=b) is a partition of the [a, b]

Set mi and Mi be a glb and lub of the set [a, b] and δi is a length of the partition I = [x_{i-1}, x_i]

Now mi = glb {f(x) | $x_{i-1} \le x \le x_i$} & Mi = lub {f(x) | $x_{i-1} \le x \le x_i$}

mi = glb {-1,1} Mi = lub {-1, 1}

` mi = -1 Mi = 1

Now L (P, f) = $\sum_{i=1}^{n}$ miδi & U (P, f) = $\sum_{i=1}^{n}$ Miδi

= lub {-(b – a)} = glb {(b – a)}

= -(b – a) = (b – a)

$$\underline{\int_a^b} fdx \ne \overline{\int_a^b} fdx$$

∴ It is not irrational of [a, b]

f∉ R [a, b]

Theorem 4: set f be a founded function defined on [a, b]

Then $\underline{\int_a^b} fdx \le \overline{\int_a^b} fdx$

Proof:-

Here f be a founded function defined on [a, b]

Set P_1& P_2 are two partition of [a, b] then clearly P=P_1UP_2 is partition of [a,b] and P is a refinement of P_1& P_2

By theorem (2) we have

L (P_1, f) ≤ L (P, f) ≤ U (P, f) ≤ U (P_1, f)

&

L (P_2, f) ≤ L (P, f) ≤ U (P, f) ≤ U (P_2, f)

L (P_1, f) ≤ U (P_2, f)

$\therefore$ U (P_2, f) is an upper bounded of {L (P, f)}

But lub {L (P, f) } = $\int_{\underline{a}}^{b} fdx$

$\therefore \int_{\underline{a}}^{b} fdx \leq$ U (P_2, f)

$\int_{\underline{a}}^{\bar{b}} fdx$ is a lb of { U (P_2, f)}

But glb {U (P,f) } = $\int_{a}^{\bar{b}} fdx$

$\int_{a}^{\bar{b}} fdx \geq \int_{\underline{a}}^{\bar{b}} fdx$

$\therefore \int_{\underline{a}}^{b} fdx \leq \int_{\underline{a}}^{\bar{b}} fdx$

Theorem: (5) State and prove Darboux's theorem.

Statement: Function f is bounded on [a, b] then

For $\forall \varepsilon>0, \exists \delta>0$ such that

U (p, f) < $\int_{a}^{\bar{b}}$ f(x) dx + ε

L (p, f) < $\int_{\underline{a}}^{b}$ f(x) dx - ε

Where $\|P\|<\delta$ for any partition P of [a. b]

OR

Riemann Integration

Unit 1(a)

Let function f is bounded on [a, b] then

$$\lim_{\|P\|\to 0} \textbf{U (P, f)} = \int_{a}^{\bar{b}} f(x)dx,\ \lim_{\|P\|\to 0} L(P,\text{f}) = \int_{\underline{a}}^{b} f(x)dx$$

Proof: Let ε> 0 is given for partitions P_1 and P_2 of [a, b]

$\therefore$ We have

$$\int_{\underline{a}}^{b} f(x)dx - \frac{\varepsilon}{3} \le L(P_1, f) \le U(P_2, f) \le \int_{a}^{\bar{b}} f(x)dx + \frac{\varepsilon}{3} \quad \text{……. (A)}$$

Let $P_0 = P_1 \cup P_2$

$\therefore$ P_0 is finer than P_1 and P_2

$\therefore$ L (P_1, f) ≤ L (P_0, f) ≤ U (P_0, f) ≤ U (P_2, f)

$\therefore$ L (P_1, f) ≤ L (P0, f) $\Rightarrow$ - L (P_1, f) $\ge$ -L (P_0, f) ……. (1)

And also U (P_2, f) $\ge$ U (P_0, f) ……. (2)

By adding appropriate side of (1), (2), we get

U (P_0, f) - L (P_0, f) ≤ U (P_2, f) - L (P_1, f) ……. (3)

Now from result (A) we have

$$\text{U (}P_2\text{, f) – L (}P_1\text{, f)} \le \int_{a}^{\bar{b}} f(x)dx - \int_{\underline{b}}^{b} f(x)dx + \frac{2\varepsilon}{3} \quad \text{……. (4)}$$

By putting value of equation (4) in equation (3), we get,

$$\text{U (}P_0\text{, f) – L (}P_0\text{, f)} \le \int_{a}^{\bar{b}} f(x)dx - \int_{\underline{b}}^{b} f(x)dx + \frac{2\varepsilon}{3} \quad \text{……. (5)}$$

Let assume that in partition P_0, there are n points and $\delta = \dfrac{\varepsilon}{3K(n-1)}$,

where K = M – m

$\therefore$ For partition where |P|<δ, we have

U (P, f) – L (P, f) ≤ U (P_0, f) – L (P_0, f) + K (n – 1) |P|

$\therefore$ U (P, f) – L (P, f) < $\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx + \dfrac{2\varepsilon}{3} + K(n-1)\delta (\because |P| < \delta and \;\; eq.(5))$

$$< \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx + \frac{2\varepsilon}{3} + K(n-1)\frac{\varepsilon}{3K(n-1)}$$

$\therefore$ U (P, f) – L (P, f) < $\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx + \varepsilon$

$$\therefore \left[U(P,f) - \overline{\int_a^b} f(x)dx\right] + \left[\underline{\int_a^b} f(x) - L(p,f)\right] < \varepsilon$$

$\therefore$ Every partition P of [a, b] such that |P|<δ,

We have 0 ≤ U (P, f) - $\overline{\int_a^b} f(x)dx < \varepsilon \Rightarrow U(P,f) < \overline{\int_a^b} f(x)dx + \varepsilon$

$$\therefore \lim_{\|P\|\to 0} U(P,f) = \overline{\int_a^b} f(x)dx$$

Similarly $0 \le \underline{\int_a^b} f(x)dx - L(P,f) < \varepsilon \Rightarrow L(P,f) > \underline{\int_a^b} f(x)dx - \varepsilon$

$$\therefore \lim_{\|P\|\to 0} L(\mathrm{P},f) = \underline{\int_a^b} f(x)dx$$

(7) Define : Riemann sum and Riemann integral (second definition)

Let f be a founded function defend on [a, b] and P = $\{x_0, x_1 \ldots . x_n\}$ is any partition of [a, b] then for i^{th} sub interval [x_{i-1}, x_i] of [a, b] and $x_{i-1} \le t_i \le x_i$ the sum

$\sum_{i-1}^{n} f(t_i)\delta_i$ is defined this sum is called **Riemann sum** it is denoted by S (P, f)

thus S (P, f) = $\sum_{i=1}^{n} f(t_i)\delta_i$

If $\lim_{\|P\| \to 0}$ S (P, f) exists them f is called **Riemann integrable** over [a, b] and this limit is called **Riemann integral**

It is denoted by $\int_a^b f dx$. Thus $\lim_{\|P\| \to 0}$ S (P, f) = $\int_a^b f dx$

Theorem 6: prove that first definition of Riemann integral ⇔ second definition of Riemann integral or set f is a bounded function defined on [a, b], the necessary and sufficient condition for f such that.

$$\underline{\int_a^b} f dx = \overline{\int_a^b} f dx = \int_a^b f dx \text{ is } \lim_{\|P\| \to 0} S(P, f) = \int_a^b f dx$$

Proof:-

Here f is a bounded function defined on [a, b]

(⇒) We will prove that

$$\underline{\int_a^b} f dx = \overline{\int_a^b} f dx = \int_a^b f dx \text{ if f } \lim_{\|P\| \to 0} S(P, f) = \int_a^b f dx$$

Suppose $\underline{\int_a^b} f dx = \overline{\int_a^b} f dx = \int_a^b f dx$ ------(1)

By Darbousx's theorem $\forall \varepsilon > 0$ there exist $\delta > 0$ ∍

U (P, f) < $\underline{\int_a^b} f dx + \varepsilon$ and

L (P, f) > $\int_a^{\bar{b}} f dx$ - ε , where ||P||<δ

From (1)

U (P, f) < $\int_a^b f dx$ + ε& L (P, f) > $\int_a^b f dx$ - ε

Now, for ∀ i = 1….n, i^{th} subinterval of [a, b]

if $x_{i-1} \le t_i \le x_i$

Then S (P, f) = $\sum_{i=1}^{n}$ $f(t_i)\,\delta_i$ and

L (P, f) ≤ S (P, f) ≤ U (P, f)

$\therefore \int_a^b f dx$ - ε < L (P, f) ≤ S (P, f) ≤ U (P, f) < $\int_a^b f dx$

$\therefore \int_a^b f dx$ - ε < S(P, f) < $\int_a^b f dx$ + ε

∴ -ε < S(P, f) - $\int_a^b f dx$ < ε

|S(P, f) - $\int_a^b f dx$ |< ε

Thus, ∀ε> 0 ∃ δ> 0 ∍

|||P|| - 0|<δ (∵ ||P||<δ) = |S (P, f) - $\int_a^b f dx$ | < ε)

∴ By definition of limit

$\lim_{\|P\|\to 0}$ S (P, f) = $\int_a^b f(dx)$

∴ Condition is necessary

(⇐) suppose $\lim_{\|P\|\to 0}$ S (P, f) = $\int_a^b f dx$

∴ By define of limit

$\therefore \forall \varepsilon > 0 \ \exists \ \delta > 0 \ \ni$

$(\because \, | \|P\| - 0| < \delta \ \ (\|P\| < \delta) \Rightarrow |S(P, f) - l| < \frac{\in}{2}$

$\Rightarrow |S\,(P, f) - l| < \frac{\in}{2}$

$\Rightarrow l - \frac{\in}{2} < S\,(P, f) < l + \frac{\in}{2}$

Now, we know that partition P , $(\|P\| < \delta)$ are have

$U(P, f) - S\,(P, f) < \frac{\in}{2}$ and $S\,(P, f) - L\,(P, f) < \frac{\in}{2}$

$l - \frac{\in}{2} < S\,(P, f) < l + \frac{\in}{2}$

$U\,(P, f) - S\,(P, f) < \frac{\in}{2}$; $S\,(P, f) - l\,(P, f) < \frac{\in}{2}$

$U\,(P, f) - \frac{\in}{2} < S\,(P, f)$; $S(P, f) < l\,(P, f) + \frac{\in}{2}$

$l - \frac{\in}{2} < l\,(P, f) + \frac{\in}{2}$; $U\,(P, f) < l + \varepsilon$

$l - \varepsilon < l\,(P, f)$, ; $U\,(P, f) - \frac{\in}{2} < l + \frac{\in}{2}$

; $U\,(P, f) < l + \varepsilon$

$\therefore L\,(P, f) \le U\,(P, f)$

$l - \varepsilon < L\,(P, f) - < l + \varepsilon$,

$|L\,(P, f) - l| < \varepsilon$, $|U\,(P, f) - l| < \varepsilon$

Thus, $\forall \varepsilon > 0$ f $\delta > 0$, $\ni | \|P\| - 0| < \delta$, $(\|P\| < \delta)$

$\Rightarrow |L\,(P, f) - l| < \varepsilon$ and $|U\,(P, f) - l| < \varepsilon$

$\therefore \lim_{\|P\| \to 0} L\,(P, f) = l$ & $\lim_{\|P\| \to 0} U\,(P, f) = l$

$\therefore$ By Darboux them.

$$\therefore \int_{\underline{a}}^{b} fdx = l \;\&\; \int_{a}^{\bar{b}} fdx = l$$

$$\therefore \int_{\underline{a}}^{b} fdx = \int_{a}^{\bar{b}} fdx = \int_{a}^{b} fdx$$

$\therefore$ Condition is sufficient.

Theorem:7

State and prove necessary and sufficient condition for a founded function defined on [a, b] to be Riemann integrable over [a, b]

Or

Necessary and sufficient condition for a bounded function defined on [a, b] to be Riemann integrable over [a, b] is that $\forall \in > 0$ there exists a partition P of [a, b] such that $u(P, f) - L(P, f) < \varepsilon$ where $\|P\| < \delta$

Proof:-

Suppose bounded function f is Riemann integrable over [a, b]

$$\int_{\underline{a}}^{b} fdx = \int_{a}^{\bar{b}} fdx = \int_{a}^{b} fdx \text{ ------------- (1)}$$

Set $\varepsilon > 0$ be given by definition of lower Riemann integral and upper Riemann integral

$$\int_{\underline{a}}^{b} fdx = \text{lub}\{L(P,f)\} \;\&$$

$$\int_{a}^{\bar{b}} fdx = \text{glb}\{U(P, f)\}$$

$\therefore$ For given $\forall \varepsilon > 0$ there exist partition P_1 & P_2

L (P, f) > $\underline{\int_a^b} fdx - \frac{\in}{2}$ & U (P, f) < $\overline{\int_a^b} fdx + \frac{\in}{2}$

L (P, f) > $\int_a^b fdx - \frac{\in}{2}$ & U (P, f) < $\int_a^b fdx + \frac{\in}{2}$ ________(2)

Let P = $P_1 \cup P_2$ then P is a partition of [a, b] and P is refinement of P_1 & P_2

By the theorem (2)

L (P_1, f) ≤ L (P, f) ≤ U (P, f) ≤ U (P_1, f)

L (P_2, f) ≤ L (P, f) ≤ U (P, f) ≤ U (P_2, f) -------------- (3)

From (2) & (3)

L (P, f) > $\int_a^b fdx - \frac{\in}{2}$ & $\int_a^b fdx + \frac{\in}{2}$ > U (P, f)

By adding respectively side.

L (P, f) + $\int_a^b fdx + \frac{\in}{2} > \int_a^b fdx - \frac{\in}{2}$ + U (P, f)

U (P, f) – L (P, f) < $\int_a^b fdx + \frac{\in}{2} - \int_a^b fdx + \frac{\in}{2}$

U (P, f) – L (P, f) <ε

∴ The condition is necessary.

(⇐) Suppose ∀ε> 0 f a partition P ∃ U (P, f) – L (P, f) <ε

By the definition of low Riemann integral and Upper Riemann integral

$\underline{\int_a^b} fdx$ ≥ L (P, f)& $\overline{\int_a^b} fdx$ ≤ U (P, f)

∴ L (P, f) ≤ $\int_a^b fdx$; $\int_a^b fdx$ ≤ U (P, f)

By the theorem number (3)

$$\underline{\int_a^b} f dx \leq L(P, f) \leq U(P, f) < \overline{\int_a^b} f dx$$

$$U(P, f) - L(P, f) \leq \overline{\int_a^b} f dx - \underline{\int_a^b} f dx < \varepsilon$$

$$\therefore \overline{\int_a^b} f dx - \underline{\int_a^b} f dx > \varepsilon$$

Sine ε> 0 is arbitrary.

By talking $\lim \in > 0$ in above equation we get

$[f(x) < g(x); \lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$ ---------(A)

Again by theorem (3)

$$\underline{\int_a^b} f dx \leq \overline{\int_a^b} f dx$$

$$0 \leq \overline{\int_a^b} f dx - \underline{\int_a^b} f dx \text{ ---------(B)}$$

From (A) & (B)

$$\underline{\int_a^b} f dx - \overline{\int_a^b} f dx = 0$$

$$\therefore \underline{\int_a^b} f dx = \overline{\int_a^b} f dx$$

∴ f is Riemann Integrable.

Definition: Oscillation and Oscillatory Sum

Let M = Maximum f(x) in [a, b]

m = Minimum f(x) = [a, b]

Then M-m is called the Oscillation of the function in [a, b].

The difference U (P, f) – L (P, f), denoted by W (P, f), is called the oscillatory sum.

Theorem: (8) can be written as, "The function f is R-integrable iff

W(P, f) <ε.

Theorem:9 If a function f is continuous on [a, b] then it is Riemann integrable over [a, b]

Proof:-

Let f is a continuous function of [a, b]

∴f is uniformly continuous on [a, b]

∀ε>0 ∃ a partition P = {a = x_0 ….. x_n =b}

For i^{th} subinterval [x_{i-1}, x_i] the value of M_i – m_i is less than $\frac{\in}{b-a}$

Now U (P, f) – L (P, f) = $\sum_{i=1}^{n} M_i\delta_i - \sum_{i=1}^{n} m_i\delta_i$

$= \sum_{i=1}^{n} (M_i - m_i)\,\delta_i$

$< \sum_{i=1}^{n} \frac{\in}{b-a}\,\delta_i$

$= \frac{\in}{b-a} \sum_{i=1}^{n} \delta_i$

$= \frac{\in}{b-a}\,(b-a)$

= ε

∴U (P, f) – L (P, f) <ε

∴∀ε 0 ∃ a partition P ∍ U (P, f) – L (P, f) <ε

f in integrable function.

Theorem:10 If a function f is monotonic on [a, b] then it is Riemann integrable over [a, b]

Proof:-

Here function f is monotonic in [a, b] so, the function f is either increasing or decreasing

Case (i) suppose function f is monotonic increasing on [a, b]

$\therefore \forall \in > 0$ f a partition P = { x_0 = a …… x_n = b} such that for i^{th} subinterval

[x_{i-1}, x_i] the value of δ_i **is less than** $\frac{\in}{f(b)-f(a)}$

Now, U (P, f) – L (P, f) = $\sum_{i=1}^{n} M_i\delta_i - \sum_{i=1}^{n} m_i\delta_i$

$= \sum_{i=1}^{n} (M_i - m_i)\delta_i$

$= \sum_{i=1}^{n} \frac{\in}{f(b)-f(a)} \delta_i$

$= \frac{\in}{f(b)-f(a)} \sum_{i=1}^{n} \delta_i$ (f is increasing)

$= \frac{\in}{f(b)-f(a)} \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))$

$= \frac{\in}{f(b)-f(a)} [f(x_1) - f(x_0)] + [f(x_2) - f(x_1)] + \dots + [f(x_n) - f(x_{n-1})]$

$= \frac{\in}{f(b)-f(a)} [f(x_n) - f(x_0)]$

$= \frac{\in}{f(b)-f(a)} f(b) - f(a)$

$= \in$

$\therefore$ U (P, f) – L (P, f) $< \in$

$\forall \in > 0$ the partition of P

U (P, f) – L (P, f) $< \in$

$\therefore$ f is integrable on function

Case (2) suppose function f is monotonic decreasing on [a, b]

$\forall \in >0$ f a partition P = {a = x_0……. x_n = b} such that for i^{th}

Subinterval $[x_i, x_{i-1}]$ the value of δ_i is less than $\frac{\in}{f(a)-f(b)}$

$$U(P, f) - L(P, f) = \sum_{i=1}^{n} M_i\delta_i - \sum_{i=1}^{n} m_i\delta_i$$

$$= \sum_{i=1}^{n} (M_i - m_i)\delta_i$$

$$= \sum_{i=1}^{n} \frac{\in}{f(a)-f(b)} (M_i - m_i)$$

$$= \frac{\in}{f(a)-f(b)} \sum_{i=1}^{n} (f(x_{i-1}) - f(x_i)) \text{ (f is a decreasing)}$$

$$= \frac{\in}{f(a)-f(b)} [f(x_0) - f(x_1)] + [f(x_1) - f(x_2)] + \dots + [f(x_{n-1}) - f(x_n)]$$

$$= \frac{\in}{f(a)-f(b)} [f(x_0) - f(x_n)]$$

$$= \frac{\in}{f(a)-f(b)} f(a) - f(b)$$

$$= \in$$

$\therefore$ U (P, f) – L (P, f) $<\in$

$\forall \in > 0$ the partition of $\exists$ U (P, f) – L (P, f) $<\in$

$\therefore$f is integrable on function

Ex :12 For function f(x) = 3x + 1, x $\in$ [1, 2] show that f is R-integrable over [1, 2]

and prove that $\int_1^2$ (3x + 1) dx = $\frac{11}{2}$

Solution:-

Here f(x) = 3x+1 is increasing function in [a, b]

f(x) = 3x+1 is Riemann integrable

Now divide [1, 2] into n intervals of equal lengths.

δi = $\frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$

Partition P = {1, 1+$\frac{1}{n}$, 1 + $\frac{2}{n}$, 1 +$\frac{3}{n}$ +…..+ 1 +$\frac{n}{n}$ =2}

i^{th} subinterval [1 + $\frac{(i-1)}{n}$, 1 + $\frac{i}{n}$]

Mi = f (1 + $\frac{i}{n}$)

= 3 (1 + $\frac{i}{n}$) + 1

= 4 + $\frac{3i}{n}$

$\int_a^b$ f dx = $\lim_{n\to\infty}$ U (P, f)

= $\lim_{n\to\infty} \sum_{i=1}^{n}$ Miδi

= $\lim_{n\to\infty} \sum_{i=1}^{n}$ [4 + $\frac{3i}{n}$] $\frac{1}{n}$

$$= \lim_{n\to\infty} \left[\sum_{i=1}^{n} \frac{4}{n} + \sum_{i=1}^{n} \frac{3i}{n^2} \right]$$

$$= \lim_{n\to\infty} \left[\frac{4}{n} \sum_{i=1}^{n} i + \frac{3}{n^2} \sum_{i=1}^{n} i \right]$$

$$= \lim_{n\to\infty} \left[\frac{4}{n} n + \frac{3}{n^2} \frac{n(n+1)}{2} \right]$$

$$= \lim_{n\to\infty} \left[\frac{4}{n} + \frac{3}{n^2} n^2 \frac{(1+1/n)}{2} \right]$$

$$= \lim_{n\to\infty} \left[4 + 3 \frac{(1+1/n)}{2} \right]$$

n $\to\infty$ = $\frac{1}{n} \to 0$

$$\left[4+3\frac{(1+0)}{2}\right] = 4 + \frac{3}{2}$$

$$\therefore \int_a^b f dx = \frac{11}{2}$$

Ex 8: If f(x) = $\frac{1}{\sqrt{x}}$, x ∈ [1, d] then prove that $\int_1^4 \frac{1}{\sqrt{x}}$ dx = 2

Solution:-

f(x) = $\frac{1}{\sqrt{x}}$ x ∈ [1, 4]

$\therefore$ f is increasing function

$\therefore$ f is R- integrable over [1, 4]

Now divide [1, 4] into n intervals of equal lengths.

Lengths of each subinterval

$$\delta i = \frac{b-a}{n} = \frac{4-1}{n} = \frac{3}{n}$$

$$P = \left\{1, 1+\frac{3}{n}, 1+2\frac{3}{n}, 1+3\frac{3}{n}+ \ldots + 1+n\frac{3}{n}\right\}$$

For i^{th} interval

$$\left[1+(i-1)\frac{3}{n}, 1+\frac{i3}{n}\right] \text{ (f is increasing)}$$

$$mi = f\left(1+\frac{i3}{n}\right)$$

$$mi = \frac{1}{\sqrt{1+\frac{i3}{n}}}$$

$$\int_a^b f\,dx = \int_a^b \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{n\to\infty} L(P, f)$$

$= \lim_{n\to\infty} \sum_{i=1}^{n}$ miδi

$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3/n}{\frac{1}{\sqrt{1+\frac{i3}{n}}}}$

$= \lim_{n\to\infty} \frac{3}{n}$

$= \lim_{n\to\infty} 3\left(0+\frac{1}{n}\right)$

Set $n\to\infty \frac{1}{n} \to 0$

$= \lim_{n\to\infty} 3(0+0)$

= 0

Ex 9: the function f is defined as follows.

f(x) = $\frac{1}{a^{r-1}}; \frac{1}{a^r} < x < \frac{1}{a^{r-1}}, r = 1,2,3\ldots$

= 0 ; x = 0

Then prove that f ∈ R [0, 1] and find $\int_0^1$ **f(x) dx**

Solution:- Let f(x) = $\frac{1}{a^{r-1}}$

Take r = 1, 2, 3.... Successively,

We get, subintervals, length of subintervals and value of function in each sub intervals.

For r = 1

f (x) = $\frac{1}{a^{r-1}}; \frac{1}{a^r} < x < \frac{1}{a^{r-1}}$

= $\frac{1}{a^0}$

= 1

$\delta_1 = 1 - \frac{1}{a} = \frac{a-1}{a}$

For r = 2

f(x) = $\frac{1}{a^{r-1}} \quad ; \frac{1}{a^2} < x < \frac{1}{a}$

= $\frac{1}{a}$

$\delta_2 = \frac{1}{a} - \frac{1}{a^2} = \frac{a-1}{a^2}$

For r = 3

f(x) = $\frac{1}{a^{r-1}} \quad ; \frac{1}{a^3} < x < \frac{1}{a^2}$

= $\frac{1}{a^{3-1}}$

f(x) = $\frac{1}{a^2}$

$\delta_3 = \frac{1}{a^2} - \frac{1}{a^3} = \frac{a-1}{a^3}$

Set of all points' discontinuity of function f is $\left\{\frac{1}{a}, \frac{1}{a^2}, \frac{1}{a^3} \ldots \frac{1}{a^n}\right\}$

This set has a limit points o which is finite

$\therefore$ f is Riemann integrable on [0, 1]

Then $\int_0^1 f(x)\,dx = \lim_{n\to\infty} S(P, f)$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} f(t_i)\,\delta_i$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} f(x)\,\delta_i \qquad \text{(take ti = x)}$$

$$= \lim_{n\to\infty}\left[1\left(\frac{a-1}{a}\right)+\left(\frac{a-1}{a^2}\right)+\frac{1}{a^2}\left(\frac{a-1}{a^3}\right)+\ldots\right]$$

$$= \frac{(a-1)}{a}\lim_{n\to\infty}\left[1+\frac{1}{a^2}+\frac{1}{a^4}+\ldots\right]$$

$$= \frac{(a-1)}{a}\,\frac{a}{a^2-1} \qquad \left(s=\frac{a}{1-r} \text{ where, a = first term ,r= ratio}\right)$$

$$= \frac{(a-1)}{a}-\frac{1}{1-\left(\frac{1}{a^2}\right)}$$

$$= \frac{(a-1)}{a}\,\frac{a^2}{(a^2-1)}$$

$$= \frac{a}{(a+1)}$$

Ex:10 the function f is defined as follows:-

$$f(x)=\frac{1}{2^n};\frac{1}{2^{n+1}}<x<\frac{1}{2^n}, n=0,1,2\ldots$$

$$= 0; \quad x = 0$$

Solution:- Take n = 0, 1, 3..... Successively

We get sub intervals length of subintervals and value of function in each subinterval

For n = 0

f(x) = 1 $\frac{1}{2} < x < 1$

$\delta_1 = 1 - \frac{1}{2} = \frac{1}{2}$

For n = 1

f(x) = $\frac{1}{2}; \frac{1}{4} < x < \frac{1}{2}$

$\delta_2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

for n = 2

f(x) = $\frac{1}{4}; \frac{1}{8} < x < \frac{1}{4}$

$\delta_3 = \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$

$\therefore$ Set of all points' discontinuity of function f is $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8} \ldots\right\}$

This set has a limit point of which is finite is Riemann integrable on [0, 1].

$\int_0^1 f(x)\, dx = \lim_{n\to\infty} S(P, f)$

$= \lim_{n\to\infty}\sum_{i=1}^{n}$ f(ti) δi (take f(ti)=f(x))

$$= \lim_{n\to\infty}\left[1\left(\frac{1}{2}\right)+\frac{1}{2}\left(\frac{1}{4}\right)+\frac{1}{4}\left(\frac{1}{8}\right)+..\right]$$

$$=\lim_{n\to\infty}\left[\frac{1}{1}+\frac{1}{8}+\frac{1}{32}+\ldots\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{2}+\frac{1}{2^3}+\frac{1}{2^5}+\ldots\right]$$

$$= \frac{1}{2}\lim_{n\to\infty}\left[1+\frac{1}{2^2}+\frac{1}{2^4}+\ldots\right]$$ (s= $\frac{a}{1-r}$ where, a = first term , r= ratio)

$$=\frac{1}{2}\left(\left(\frac{1}{1-\frac{1}{4}}\right)\right)$$

$$= \frac{1}{2}\cdot\frac{4}{3}$$

$$\therefore \int_0^1 f\,dx = \frac{2}{3}$$

Ex:11 Show that f(x) = [x]; x ∈ [0, 3] is R-integrale over [0,3] and

find $\int_0^3$ [x] dx.

Solution:-

Here f(x) = [x], x∈[0, 3]

f(x) = 0 a ≤ x ≤ 1

= 1 $1 \le x \le 2$

= 2 $2 \le x \le 3$

= 3 $3 \le x \le 4$

Set of paints of discontinuity of f is { 1, 2, 3.....} which is finite

$\therefore$ f is R-integrable

$$\int_0^3 f\,dx = \int_0^1 f\,dx + \int_1^2 f\,dx + \int_2^3 f\,dx + \ldots..$$

$$= \int_0^1 0\,dx + \int_1^2 1\,dx + \int_2^3 2\,dx + \ldots..$$

$$= 0 + \left[x\right]_1^2 + 2\left[x\right]_2^3 + \ldots. +$$

$$= 0 + [\,2 - 1 + 6 - 4]$$

$$\int_0^3 f\,dx = 3$$

*** (Working rules of Riemann integration)***

Theorem:11 If f and g are R- integrable over [a, b] then (f + g) is also a R-integrable over [a, b] and $\int_b^b (f+g)dx = \int_a^b f dx + \int_a^b g dx$

Proof: Here f and g are R- integrable over [a, b]

By the de^f of R integrable

$\forall \in > 0\ \exists$ a partition of P such that

$\lim_{\|P\| \to 0}$ S (P, f) and $\lim_{\|P\| \to 0}$ S (P, g) -------(1)

And $\int_a^b f dx = \lim_{\|P\| \to 0}$ S (P, f)

And

$\int_a^b f dx = \lim_{\|P\|\to 0}$ S (P, g) ……………….. (2)

Now, for $x_{i-1} \le t_i \le x_i$ we find,

S (P, f + g) = $\sum_{i=1}^{n}$ (f + g) $(t_i)\ \delta_i$

$= \sum_{i=1}^{n} [f(t_i) + g(t_i)]\ \delta_i$

$= \sum_{i=1}^{n} f(ti)\delta i + \sum_{i=1}^{n} g\ (ti)\ \delta i$

S (P, f + g) = S (P, f) + S (P, g)

$\lim_{\|P\|\to 0}$ S (P, f + g) = $\lim_{\|P\|\to 0}$ S (P, f) + $\lim_{\|P\|\to 0}$ S (P, g) ………………(3)

From (1)

Now both limit on R.H.S. exist and limit on L.H.S. exist.

i.e. $\lim_{\|P\|\to 0}$ S (P, f + g) exist.

f + g is R- integrable over [a, b]

Also $\int_a^b$ (f + g) dx = $\lim_{\|P\|\to 0}$ S (P, f) + $\lim_{\|P\|\to 0}$ S (P, g)

$= \int_a^b f\,dx + \int_a^b g\,dx$

$\therefore \int_a^b (f + g)\,dx = \int_a^b f\,dx + \int_a^b g\,dx$

Theorem:12 if f is R- integrable over [a, b] then (c f) is also a R-integrable over [a, b] and $\int_a^b (c\,f)dx = C\int_a^b f\,dx$.

Proof:-

Here f is R integrable over [a, b]

$\therefore$ By def ignition of R- integrable $\forall \in > 0 \ \exists$ a partition of P such that

$\lim_{\|P\|\to 0}$ S (P, f) exist …………….. (1)

And $\int_a^b$ f dx = $\lim_{\|P\|\to 0}$ S (P, f)

Now for $x_{i-1} \le ti \le x_i$

We find S (P, f) = $\sum_{i=1}^{n}$ (C f(ti)δi) = C $\sum_{i=1}^{n}$ (f(ti)δi)

S (P, f) = C S(P, f)

$\lim_{\|P\|\to 0}$ S (P, f) = $\lim_{\|P\|\to 0}$ (C S (P, f))

= C $\lim_{\|P\|\to 0}$ S (P, f)

From (1)

Now both limits on R.H.S. exist

$\because$ Limit on L.H.S. exists

i.e. $\lim_{\|P\|\to 0}$ S (P, f) exists.

C is R- integrable over [a, b]

Also, $\int_a^b$ C f dx = $\lim_{\|P\|\to 0}$ C S (P, f)

= C $\lim_{\|P\|\to 0}$ S (P, f)

= C $\int_a^b$ f dx

= $\int_a^b$ C f dx = C $\int_a^b$ f dx

Theorem 13: If f $\in$ R [a, b] and a < c < b then f $\in$ R [a, c] and

g$\in$ R [c, b] Also $\int_a^b$ f dx = $\int_a^c$ f dx + $\int_c^b$ f dx

Proof:-

Here $f \in R[a, b]$

i.e. f is R-integrable over [a, b]

$\therefore \forall \in > 0 \; \exists$ a partition of P such that $P = \{a = x_0 \ldots\ldots x_n = b\}$

$U(P, f) - L(P, f) < \varepsilon$ ……..(1)

Now, $a < c < b$

Let $P_1 = P \cup \{c\}$

Then P_1 is a partition of [a, b] and P_1 is refinement of P

$L(P, f) \leq L(P_1, f) \leq U(P_1, f) \leq U(P, f)$

$U(P_1, f) - L(P_1, f) \leq U(P, f) - L(P, f)$

From

$U(P_1, f) - L(P_1, f) < \varepsilon$ ……..(2)

Let P_2& P_3 are subset of P_1 such that P_2 is a partition of [a, c] and P_3 is a partition of [a, b]

Also $P_1 = P_2 \cup P_3$

&

$U(P_2, f) + U(P_3, F) = U(P_1, f)$

$L(P_2, f) + L(P_3, f) = L(P_1, f)$

$[U(P_2, f) - L(P_2, f)] + [U(P_3, f) - L(P_3, f)] = [U(P_1, f) - L(P_1, f)]$

Now, each bracket on L.H.S. is non negative

$U(P_2, f) - L(P_2, f) \leq U(P1, f) - L(P_1, f) < \varepsilon$ ………..from (2)

$U(P_3, f) - L(P_3, f) \leq U(P1, f) - L(P_1, f) < \varepsilon$

Thus for

$\forall \in > 0 \; \exists$ a partition P_2 of [a, b] and P_3 of [c, b]

$U(P_2, f) - L(P_2, f) < \varepsilon$

$U(P_3, f) - L(P_3, f) < \varepsilon$

F R [a, c] & $f \in R[c, b]$

Again

$U(P_2, f) + U(P_3, f) = U(P_1, f) \geq \int_a^b f dx$

$\{\int_a^b f\, dx = \text{glb}\,\{U(P_1, f)\}$

$U(P_2, f) + U(P_3, f) \geq \int_a^{\bar{b}} f\, dx$

$\text{Glb}\,[U(P_2, f) + U(P_3, f)] \geq \int_a^{\bar{b}} f\, dx$

$\text{Glb}\{U(P_2, f)\} + \text{glb}\{U(P_3, f)\} \geq \int_a^{\bar{b}} f\, dx$

$\int_a^{\bar{c}} f\, dx + \int_a^{\bar{b}} f\, dx \geq \int_a^{\bar{b}} f\, dx$

But f ε R [a, b] , f ε R [a, c] , f $\in$ R [c, b]

$\int_a^{\bar{c}} fdx + \int_c^{b} fdx \geq \int_a^{b} fdx$

Similarly we can prove that

$\int_a^{c} f\, dx + \int_c^{b} f\, dx \leq \int_a^{b} f\, dx$

Thus $\int_a^{c} f\, dx + \int_c^{b} f\, dx \neq \int_a^{b} f\, dx$

Theorem:14 if a bounded function f is R-integrable over [a, b] then |f| is also R-integrable over [a, b] and bounded $|\int_a^b f\, dx \leq \int_a^b |f| dx$

Proof:-

Here f is bounded in [a, b] f positive number k $\exists |f(x)| \leq k \ \forall\ x \in [a, b]$

Also, f is Riemann integrable over [a, b]

$\therefore$ For $\forall \varepsilon > 0$ $\exists$ a partition P such that $U(P, f) - L(P, f) < \varepsilon$ …………(1)

Than clearly f is bounded in [a, b]

Let mi and Mi and mi' and Mi' are glb and lub of |f| and f in $[x_{i-1}, x_i]$ respectively.

Now $\forall\ x_1, x_2, \in [x_{i-1}, x_i]$

$||f|(x_1) - |f|(x_2)| \le ||f(x_1)| - |f(x_2)|| \le |f(x_1)| - |f(x_2)|$

Mi – mi ≤ Mi' – mi' (multiplying by δi)

Miδi - miδi ≤ Mi'δi – mi'δi

$$\sum_{i=1}^{n} M_i\delta_i - \sum_{i=1}^{n} m_i\delta I \le \sum_{i=1}^{n} M_i'\delta_i - \sum_{i=1}^{n} m_i'\delta_i$$

U (P, |f|) – L (P,|f|) ≤ U (P, f) – L (P, f)

∴ U (P, |f|) – L (P, |f)|<ε from (1)

Thus , for ε> 0 ∃ a partition such that U (P, |f|) – L (P,|f|) <ε

|f| is Riemann integrable over [a, b]

∀xε [a, b]

Now f(x) ≤ |f(x)| and-f(x) ≤ |f(x)|

$$\therefore \int_a^b f(x)dx \le \int_a^b |f(x)|dx \text{ and}$$

$$-\int_a^b f(x)\,dx \le \int_a^b |f(x)|dx$$

$$|\int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx$$

$$\therefore |\int_a^b f\,dx \le \int_a^b |f|\,dx$$

Theorem:15 if f and g are R-integrable over [a, b] then (f. g) is also a R-integrable over [a, b]

Proof:-

Here f & g are bounded in [a, b]

There exists a positive number k & k' such that

$|f(x)| \le k$ & $|g(x)| \le k'$; $\forall x \in [a, b]$

Now $\forall x \in [a, b]$

$\therefore |(f\bullet g)(x)| = |f(x) \bullet g(x)| \leq |f(x)||g(x)| \leq k - k'$

$(f\bullet g)$ is a bounded function in [a, b]

Now f & g are Riemann integrable over [a, b]

$\therefore \forall \in > 0$ there exists a partition p such that

$U(P, f) - L(P, f) < \frac{\in}{2k'}$ ....(1)

$U(P, g) - L(P, g) < \frac{\in}{2k}$

mi , Mi and mi' &Mi' and m;" &Mi" are glb and lub of $|f . g|$

& f & g in [xi-1, xi] respectively.

Now for $\forall x_1, x_2 \in [x_{i-1}, x_i]$

$|(fg)x_1 - (fg)x_2| = |(f(x_1)g(x_1) - f(x_1)g(x_2) + f(x_1)g(x_2) - f(x_2)g(x_2)|$

$= |(f(x_1)(g(x_1) - g(x_2) + g(x_2)f(x_1) - f(x_2))|$

$\leq k|g(x_2\text{-}1) - g(x_2)| + k'|f(x_1) - f(x_2)|$

Mi – mi ≤ k (Mi" – mi") + k' (Mi' – mi') (multiplying by δi)

Miδi - miδi ≤ k (Mi" – mi")δi + k' (Mi'δi – mi'δi)

$\sum_{i=1}^{n}$ Miδi - $\sum_{i=1}^{n}$ miδi≤ k [$\sum_{i=1}^{n}$ M"iδi - $\sum_{i=1}^{n}$ mi"δi] + k' [$\sum_{i=1}^{n}$ Mi'δi - $\sum_{i=1}^{n}$ mi'δi]

$U(P, f\cdot g) - L(P, f\cdot g) \leq k[U(P, g) - L(P, g)] + k'[U(P, f) - L(p, f)]$

$\mathbf{U(P, f\cdot g) - L(P, f\cdot g) < K \cdot \frac{\in}{2k} + k' \frac{\in}{2k'}}$

$U(P, f\cdot g) - L(P, f\cdot g) < \in$

Thus, for $\in > 0$ there exists a partition f such that

$U(P, f\cdot g) - L(P, f\cdot g) < \in$

Thus $|f\cdot g|$ is Riemann integrable on [a, b]

Theorem 16 : If bounded function $f \in R_{[a,b]}$ then $f^2 \in R_{[a,b]}$

Proof: Since, f is bounded on [a, b], therefore $\exists$ M > 0,

such that $|f(x)| \le M$, $\forall x \in [a, b]$

Now, since $f \in R_{[a, b]}$, $|f| \in R_{[a, b]}$ therefore for $\varepsilon > 0$ $\exists$ a partition P of [a, b] such that $U(P, |f|) - L(P, |f|) < \frac{\varepsilon}{2M}$ ....(1)

Again, since $|f^2(x)| = |f(x)|^2 \le M^2$

$\therefore f^2$ is also bounded.

If M_i , m_i be the bounds of $|f|$ and M'_i, m'_i be the bounds of f^2 in $[x_{i-1}, x_i]$ then $M'_i = M^2_i$, $m'_i = m^2_i$ and also

$$U(P, f^2) - L(P, f^2) = \sum_{i=1}^{n}(M'_i - m'_i)\Delta x_i$$

$$= \sum_{i=1}^{n}(M_i^2 - m_i^2)\Delta x_i$$

$$= \sum_{i=1}^{n}(M_i - m_i)(M_i + m_i)\Delta x_i$$

$$\therefore U(P, f^2) - L(P, f^2) \le 2M \left\{\sum_{i=1}^{n}(M_i - m_i)\Delta x_i\right\}$$

$$= 2M \left\{\sum_{i=1}^{n} M_i \Delta x_i - \sum_{i=1}^{n} m_i \Delta x_i\right\}$$

$$= 2M \{ U(P, |f|) - L(P, |f|)\}$$

$$\therefore U(P, f^2) - L(P, f^2) < 2M \frac{\varepsilon}{2M} \quad (\because \text{from (1)})$$

$\therefore U(P, f^2) - L(P, f^2) < \varepsilon$

$\therefore f^2 \in R_{[a,b]}$