SHREE H.N.SHUKLA GROUP OF COLLEGES

M.Sc. (Mathematics)
Semester-IV

IMP Questions of Number Theory-2

Q-1 Short Questions

1. Find the value of r_0, r_1, r_2, and r_3 for the continued fraction expansion $\langle 1, 1, 1, 1, 2\rangle$.
2. Find the general solution (if any) of the Diophantine equation $2x + 9y = 18$.
3. Find first four positive solution of $x^2 - 8y^2 = 1$.
4. Find two Primitive Pythagorean triplet (x, y, z) for which $z < 59$.
5. Find two positive integers n such that $1 + 2 + 3 + 4 + \cdots + n$ is a perfect square.
6. Define with examples: (a) Quadratic irrational and (b) Pell's equation.
7. Define: (a) Primitive Pythagorean triplet (b) Simple Continued fraction expansion.
8. Give the definition of Diophantine equation.
9. Express the rational numbers $\frac{101}{7}$ and $\frac{1437}{11}$ as simple continued fractions.
10. Express $\frac{2018}{17}$ as a simple continued fraction.
11. Find the continued expansion of $\frac{\sqrt{5}+1}{2}$ and $\frac{\sqrt{5}-1}{2}$.
12. Show that $\gcd(x, y) = \gcd(y, z)$, where (x, y, z) is a Pythagorean Triplet.
13. Write down all Farey Fractions between 0 and 1 up to 7^{th} row.
14. Find the value of $\langle 1, 1, 2, 2, 2, \ldots \ldots\rangle$ and $\langle -2, 2, 4, 3, 3, \ldots \ldots\rangle$.
15. Find atleast three positive solutions of the equation $x^2 - 2y^2 = 1$.
16. If r and s are positive integers and t is a rational solution of $x^r = s$ then t must be____. Justify your answer.
17. Show that there are infinitely many solutions (x, y) of $x^2 - dy^2 = 1$ in which k/y for $d > 1$ is not a perfect square and $k \geq 1$.
18. If $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive Farey fractions in the n^{th} row and $\frac{a}{b}$ is less than $\frac{c}{d}$ then $\frac{a}{b}$ and $\frac{a+c}{b+d}$ are consecutive Farey fractions in the ______ row.
19. If $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive Farey fractions in the n^{th} row then $\left|\frac{a}{b} - \frac{a+c}{b+d}\right| \leq$______.
20. If the simple continued fraction expansion of θ is finite then θ must be a_____ number.
21. If θ is an irrational number and $\frac{a}{b}$ is a rational number such that $b > 0$ and $\left|\theta - \frac{a}{b}\right| < \frac{1}{2b^2}$ then $\frac{h_n}{k_n} =$ _______ for some n.
22. If continued fraction expansion of an irrational θ is periodic and θ' lies between -1 and 0 then continued fraction expansion of θ is ________.
23. If θ is an irrational, $\frac{a}{b}$ is a rational number such that $\left|\theta - \frac{a}{b}\right| < \left|\theta - \frac{h_n}{k_n}\right|$ for some $n \geq 0$ then b is greater than _______.
24. The Diophantine equation $ax + by = c$ has a solution if and only if ________ divides c.

Q-2 Long Questions.

1. State and prove Hurwitz Inequality for simple continued fractions.
2. Prove that for an irrational number x the infinite continued expansion is always unique.
3. If $a_0 a_1 a_2, \ldots a_n \ldots$ is sequence of integers with $a_i \geq 1$; for $i = 1, 2, 3, \ldots$. Then show that $|xk_n - h_n| < \left|\frac{1}{k_{n+1}}\right|$; $\forall n$, where x is an irrational number.

4. Suppose $u \neq 0$ is an integer and $v > 1$ is not a perfect square with $|u| < \sqrt{v}$. If (a,b) is the positive solution of $x^2 - vy^2 = u$ then $a = h_n$ and $b = k_n$, for some n provided $(a, b) = 1$.
5. Find three positive solutions of the equation $x^2 - 3y^2 = 1$.
6. Justify: Is $x^2 - 15y^2 = -1$ has a solution in integers?
7. Suppose $\langle a_0, a_1, a_2, \dots, a_n, \dots \rangle$ be an infinite sequence of integers with $a_i \geq 1$; for $i = 1, 2, 3, \dots, n$. Then prove that the subsequences r_{2j} and r_{2j-1} both converge to the same point, where h_j, k_j and r_j are defined as usual.
8. Prove that if θ is an irrational number and suppose for some rational number $\frac{a}{b}$ for $b > 0$ and $(a, b) = 1$ with $|\theta b - a| < |\theta k_n - h_n|$; for some n then $b \geq k_{n+1}$.
9. Prove that the equation is $30x^2 - 14y^2 = 18$ does not have solution in integer.
10. Write an algorithm to find the sequence of integers $a_0 a_1 a_2, \dots a_n \dots$ when an irrational number is given and then explain with an example.
11. Prove that the value of $f(x) = x^4 + x^3 + x^2 + x + 1$ is a perfect square only for $x = -1, 0, 3$ otherwise $f(x)$ is not a perfect square.
12. If $x > 1$ and -1< $x' < 0$ the show that the continued fraction expansion of x is purely periodic provided x is a quadratic irrational.
13. Show that if the triplet (x, y, z) is a Primitive Pythagorean triplet then there exists r and s such that $r > s \geq 1$. $(r, s) = 1$ and r is even then s is odd and vice-versa.
14. State and prove the necessary and sufficient condition under which the continued Farey expansion of a quadratic irrational is purely periodic.
15. If θ irrational and $\theta = \langle a_0, a_1, a_2, \dots, a_n, \dots \rangle$ then prove that $k_n < \theta_n k_{n-1} + k_{n-2} < k_{n+1}$ for all $n \geq 0$
16. Prove that π is irrational using elementary method.
17. Prove that for each $n > 0$ there is a polynomial $f_n(x)$ of degree n, leading coefficient 1 and with integer coefficients such that $f_n(2\cos\theta) = a\cos n\theta$.
18. Suppose θ is an irrational and $\frac{a}{b}$ is a rational number such that $\left|\theta - \frac{a}{b}\right| < \frac{1}{2b^2}$ then prove that $\frac{a}{b} = r_n$ for some n.
19. Find general solutions (if any) of the following Diophantine equation:
    (1) $2x + 9y = 11$
    (2) $100x + 101y = 2018$.
20. Find the value of following continued fractions:
    (1) $\langle 1, 1, 1, 1, 1 \rangle$
    (2) $\langle 0, 2, 2, 2, 2, 2, \dots \dots \rangle$
    (3) $\langle 1, 3, 3, 3, 3, \dots \dots \rangle$
    (4) $\langle -1, 1, 1, 1, 1, 1, \dots \dots \rangle$
21. Suppose θ is an irrational number whose continued fraction expansion is periodic. Prove that θ is quadratic irrational.
22. Prove that there are infinitely many positive integers n such that $1 + 2 + 3 + 4 + \cdots + n = m^2$ for some integer m.
23. If $\frac{h_j}{k_j}$ denotes j[th] convergent of an irrational number θ then prove that for all $n \geq 1$
    (1) $|\theta k_n - h_n| < |\theta k_{n-1} - h_{n-1}|$
    (2) $\left|\theta - \frac{h_n}{k_n}\right| < \left|\theta - \frac{h_{n-1}}{k_{n-1}}\right|$
24. If θ is an irrational number then prove that there are infinitely many rational number $\frac{a}{b}$ such that $\left|\theta - \frac{a}{b}\right| < \frac{1}{2b^2}$.
25. Suppose $f(x) = a_n X^n + a_{n-1} X^{n-1} + \dots \dots + a_0$ is a polynomial of degree n, with integer coefficients and suppose $\frac{a}{b}$ is a rational number with $b > 0, (a, b) = 1$ and $\frac{a}{b}$ is a root of $f(x)$. Prove that b divides a_n and a divides a_0. Deduce that if $a_n = 1$ then $\frac{a}{b}$ must be an

integer and also deduce that if an integer x has a rational n^{th} root then it must be an integer.

26. If θ is a quadratic irrational such that (i) $\theta > 1$ (ii) $-1 < \theta' < 0$ then prove that continued fraction expansion of θ is purely periodic.
27. Prove that $x^2 = y^3 + 7$ has no solution in integers.
28. Prove that there are infinitely many positive integers n such that $n^2 + (n+1)^2$ is a perfect square.
29. Suppose $a_{0,} a_{1,} a_2, \dots, a_n$ are positive integers and $\frac{h_n}{k_n}$ are the n^{th} convergent corresponding ton this integers then prove that for all $n \geq 1$, $h_n k_{n-2} - h_{n-2} k_n = (-1)^n$.
30. Prove that θ is a rational multiple of π and $cos\theta$ is rational then $|\cos\theta|$ cannot be different from $0, \frac{1}{2}$ and 1.