SHREE H. N. SHUKLA COLLEGE OF SCIENCE

M.Sc.(Mathematics) Sem-1

Prelims Test

MATH.CMT-1004 : Theory of Ordinary Differential Equations

Time : 2.5 Hours | Total Marks : 70

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1. Answer any seven: $7 \times 2 = 14$

(a) Define linear differential equation and linear homogeneous differential equation with example of each.

(b) State Legendre Polynomial.

(c) Write the system of differential equation $y_1' = y_2 + e^t$, $y_2' = y_1 + e^{-t}$ in the matrix form.

(d) If $y_1,\ y_2$ are solutions of $(1-x^2)y'' - 2xy' + p(p-1)y = 0$ with initial conditions $y_1(0) = 0,\ y_1'(0) = -1,\ y_2(0) = 1,\ y_2'(0) = 0$ then the wronskian of $y_1,\ y_2$ at 1/2 is ....

(e) Prove that for every $n \times n$ real matrix, e^A is invertible and $(e^A)^{-1} = e^{-A}$.

(f) State existance and uniqueness theorem for n^{th} order scalar linear differential equations.

(g) Define Wronskian of n functions which are $(n-1)$ tiens differentiable on I.

(h) State first and second Fundamental Theorem of Calculus.

(i) State and prove 1^{st} shifting property in Laplace Transform.

2. Answer any two: $2 \times 7 = 14$

(a) Let $p, q : I \to \mathbb{R}$ be a continuous, $t_0 \in I,\ y_0 \in \mathbb{R}$. Then prove that the IVP: $y' + p(t)y =$

$q(t),\ y(t_0) = y_0$ has a unique solution $u(t) = y_0 e^{-P(t)} + e^{-P(t)} \int_{t_0}^{t} e^{p(r)} q(r) dr$, defined on the whole of I, where $P : I \to \mathbb{R}$ is defined by $P(t) = \int_{t_0}^{t} p(r) dr,\ \forall t \in I$.

(b) Show that $\phi(t) = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}$ is a fundamental matrix of $y' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ on $\mathbb{R}$.

(c) Compute the first five terms of the series expansion at zero of the solution of the Legendre's equation $(1 - t^2)y'' - 2ty' + \alpha(\alpha + 1)y = 0$, where α is a constant.

3. All are compulsory: $2 \times 7 = 14$

(a) Find the solution of the IVP : $y' = \begin{pmatrix} 3 & 3 \\ -5 & 3 \end{pmatrix} y + \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix},\ y(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(b) Prove that the eigen vector corresponding to the distinct eigen values of a $n \times n$ matrix A are linearly independent in $\mathbb{K}^n$.

OR

(a) Let $A(t)$ be a continuous $n \times n$ matrix on I, $\phi(t)$ be a fundamental matrix of $y' = A(t)y$ on I and C be a $n \times n$ non-singular constant matrix. Then prove that $\phi(t) \cdot C$ is a fundamental matrix of $y' = A(t)y$ on I.

(b) Let A be a constant 2×2 complex matrix then prove that there exists a constant 2×2 non-singular matrix B such that $B^{-1}AB$ has the following forms:

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \text{ and } \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}.$$

4. All are compulsory: $2 \times 7 = 14$

(a) Prove that if $\alpha = 2m$, where $m \geqslant 0$ is an integer then a second linearly independent solution of the legendre equation $(1 - t^2)y'' - 2ty' + \alpha(\alpha + 1)y = 0$ valid in a nbhd of 0 can be expressed in the form of power series converges for $|t| < 1$.

(b) Show that $ty'' + y' + y = 0$ has only one solution of the form $|t|^z \sum_{k=0}^{\infty} c_k t^k,\ c_0 = 1$ is an excluded nbhd of 0.

(c) Locate and classify all singular points of $(t - 1)^3 y'' + 2(t - 1)^2 y' - 7ty = 0$.

5. Answer any two: $2 \times 7 = 14$

(a) Solve $y'' - 3y' + 2y = 4e^{2t}$ with $y(0) = -3,\ y'(0) = 5$ using Laplace Transform.

(b) Find $L^{-1}\left(\dfrac{3z+7}{z^2 - 2z - 3}\right)$

(c) State and prove Laplace Transform of Integral.

(d) Find $L\left(t^n e^{ct}\right)$

☻ *BEST OF LUCK* ☻