Shree H. N. Shukla College of Science

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T.Y.B.SC. SEM-VI (CBCS)

SUBJECT: Mathematics

PAPER: 601

Unit: 5

Residue and Poles

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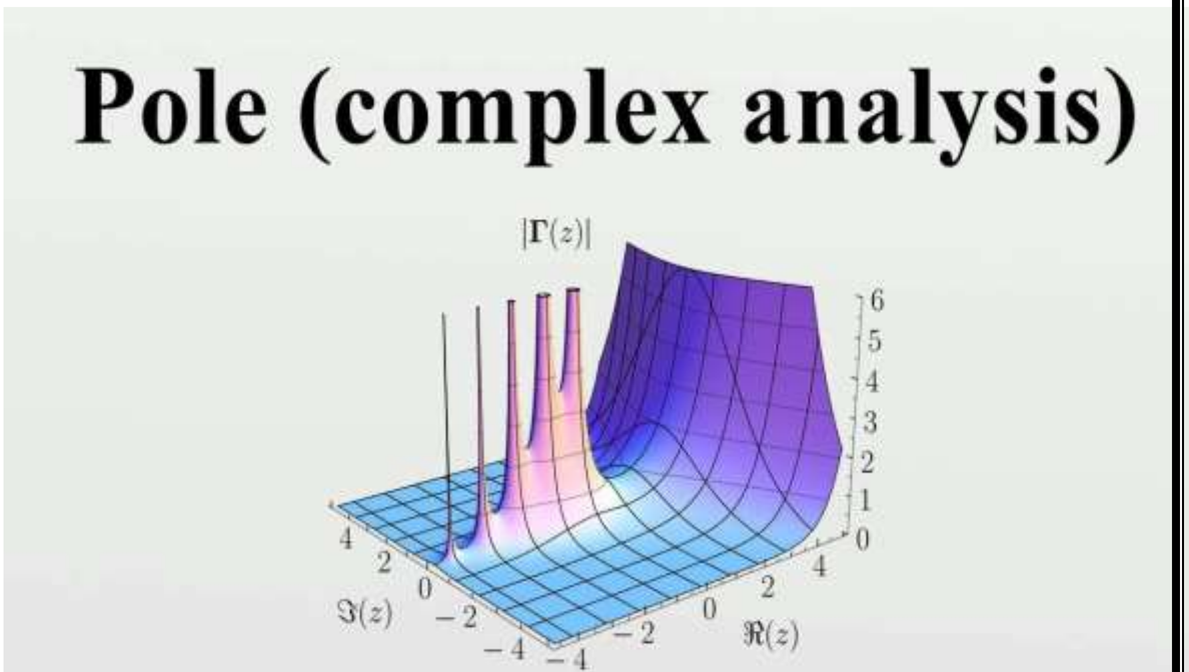
TITLE EXPLANATION

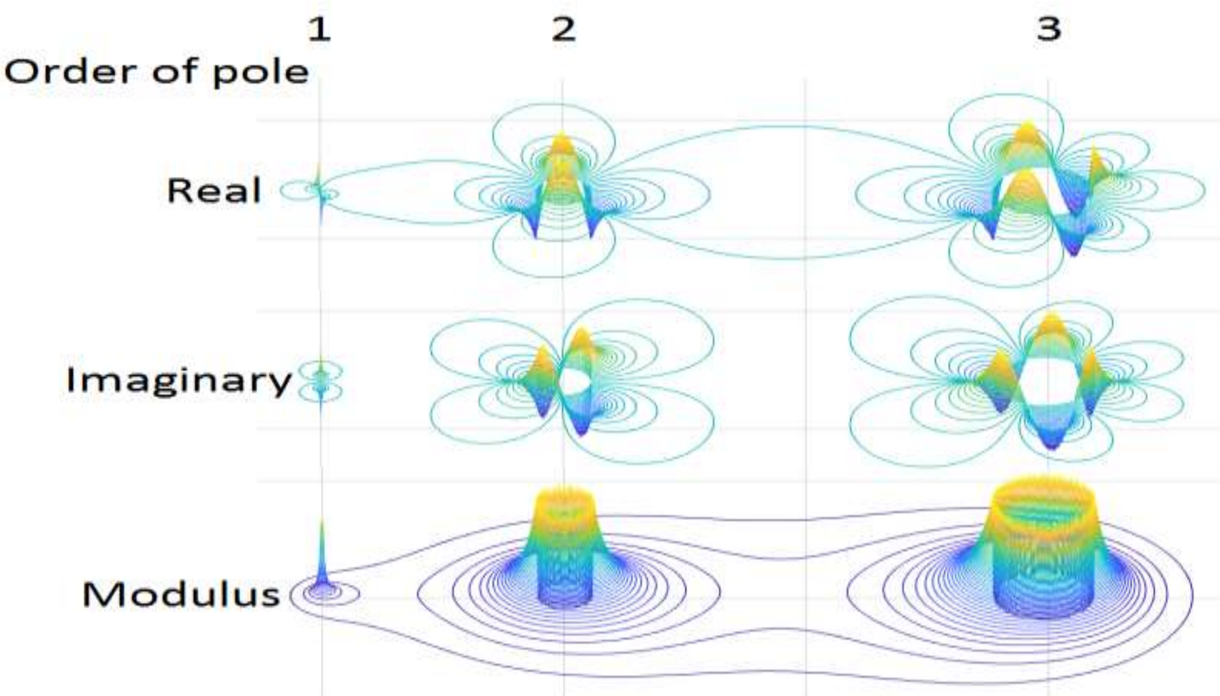
⊕ Residue:

- ✓ In general, something that remains after a part is taken, separated, or designated or after the completion of a process: remnant, remainder: such as. The part of a testator's estate remaining after the satisfaction of all debts, charges, allowances, and previous devises and bequests.
- ✓ In mathematics, more specifically complex analysis, the **residue** is a complex number proportional to the contour integral of a meromorphic function along a path enclosing one of its singularities.

⊕ **Pole:**

- ✓ In **complex analysis** (a branch of mathematics), a **pole** is a certain type of singularity of a function, nearby which the function behaves relatively regularly, in contrast to essential singularities, such as 0 for the logarithm function, and branch points, such as 0 for the **complex** square root function.


Pole (complex analysis)
|Γ(z)|
ℑ(z)
ℜ(z)


1
2
3
Order of pole
Real
Imaginary
Modulus

Trailer of Topic

- Singular point
- Zeros of complex function
- Pole,Simple pole,m^{th} order pole
- Residue of function
- Cauchy residue theorem
- Definite integral of trigonometric functions
- Evaluation of improper real integrals

LEARNING OUTCOME

- ☑ This capstone has focused on how to use complex analysis to evaluate various definite integrals in the real plane.
- ☑ Not only does this make computing certain integrals easier, but it also allows us to evaluate integrals of functions whose anti-derivative is unknown or impossible to find.
- ☑ Aside from evaluating integrals in the real plane, the amazing result of the Residue Theorem is the ability to evaluate contour integrals such that non-analytic points lie inside the closed contour.
- ☑ The Residue Theorems included in this capstone are a small sample of all the Residue Theorems.
- ☑ The basic idea behind each Residue Theorem is the same, but each theorem holds its own power and beauty.
- ☑ Without the brilliant minds that contributed to the Residue theorems, the world would not be where it is today.

♣ Definition: Singular Point

If the complex function f(z) is an analytic everywhere in nbhd of z_0, except z_0, then z_0 is called Singular point.

Example;

(i) $f(z) = \frac{1}{(z-1)(z-2)}$

$\therefore$ z_0=1 & z_0=2 are Singular point.

(ii) $f(z) = \frac{1}{z(z-2)(z-3)(z-5)}$

$\therefore$ z_0=0, z_0=2, z_0=3 & z_0=5 are Singular point.

♣ Definition: Isolated Singular Point

If the complex function f(z) is an analytic everywhere in every nbhd of z_0, except z_0. Then z_0 is called Isolated Singular point.

Example;

(i) $f(z) = \frac{1}{z}$

$\therefore$ z_0=0 is Isolated Singular point.

(ii) $f(z) = \frac{1}{(z-1)}$

$\therefore$ z_0=1 is Isolated Singular point.

Sr. No.	Question	Answer
1	If the complex function f(z) is an analytic everywhere in nbhd of z_0, except z_0, then z_0 is called………	**Singular point**
2	Write the Isolated singular point for function $f(z) = \frac{1}{(z-5)}$	**z_0=5**

♣ Definition: Zeros of Complex function

- Let f(z) be an analytic everywhere in nbhd of z_0, then by Taylor's expansion,

$$f(z) = f(z_0) + \frac{(z-z_0)}{1!}f'(z_0) + \frac{(z-z_0)^2}{2!}f''(z_0) + \dots\dots\dots\dots\dots$$

If;

$$f(z_0) = f'(z_0) = f''(z_0) = \dots\dots\dots\dots\dots. = f^{m-1}(z_0) = 0$$

But

$$f^m(z_0) \neq 0$$

Then z_0 is said to be m^th^ order zero of f(z).

- In particular m=1,

That means, $f(z_0) = 0 \ but \ f'(z_0) \neq 0$

Then z_0 is called 1^st^ order zero of f(z) OR Simple zero of f(z).

♣ Definition: Pole

- Let f(z) be an analytic everywhere in nbhd of z_0, then by Laurent's expansion of f(z) at z_0,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

where,

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz \;\; \& \;\; b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} dz$$

- Here 2nd term of R.H.S. of Laurent's expansion;

  That means, $\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ is called Principle part of Laurent's expansion.

- If Principle part of Laurent's expansion contain finite term;

  That means,

  $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \; \dots \dots \dots + \frac{b_m}{(z-z_0)^m}$

  Then z_0 is called m^{th} order pole of f(z).

- In particular m=1,

  $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{(z-z_0)}$

  Then z_0 is called 1^{st} order pole OR Simple pole.

Sr. No.	Question	Answer
1	2nd term of R.H.S. of Laurent's expansion is called……..	**Principle part of Laurent's expansion**
2	If Principle part of Laurent's expansion contain finite m term then it is said to be……..	**m th order pole of f(z)**
3	If Principle part of Laurent's expansion contain only first term then it is called…….	**Simple pole**
4	Write down the formula of Laurent's expansion.	$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$

♣ Definition: Residue of f(z)

Let f(z) is an analytic everywhere in nbhd of z_0, except z_0. Then the coefficient of $\frac{1}{z-z_0}$ in Laurent's expansion is called Residue of f(z) at pole z_0 and it is denoted by

$$b_1 = Res(f(z), z_0) = \frac{1}{2\pi i}\int_C f(z)dz$$

EXAMPLE-1:

Obtain the formula for finding residue of f(z) at Simple pole.

SOLUTION:

Let f(z) is Simple pole.

By definition of Simple pole,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{(z - z_0)}$$

$$(z - z_0) f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{n+1} + b_1$$

Now, we taking limit both sides;

$$\therefore \lim_{z \to z_0} (z - z_0) f(z) = \lim_{z \to z_0} \sum_{n=0}^{\infty} a_n (z - z_0)^{n+1} + b_1$$

$$\therefore \lim_{z \to z_0} (z - z_0) f(z) = 0 + b_1$$

$$\therefore b_1 = Res(f(z), z_0) = \lim_{z \to z_0} f(z)(z - z_0)$$

EXAMPLE-2:

Find the residue of $f(z) = \frac{z+2}{(z-1)(z-2)}$ at Simple pole.

SOLUTION:

Here, $f(z) = \frac{z+2}{(z-1)(z-2)}$

Pole is (z-1)(z-2) $\Rightarrow$ z_0=1 and z_0=2

Now,

Residue of f(z) at z_0=1

$$\therefore b_1 = Res(f(z), z_0) = \lim_{z \to z_0} f(z)(z - z_0)$$

$$\therefore b_1 = Res(f(z), 1) = \lim_{z \to 1} \frac{z+2}{(z-1)(z-2)} (z - 1) = \frac{3}{-1} = -3$$

Residue of f(z) at z_0=2

$$\therefore\ b_1 = Res(f(z),2) = \lim_{z\to 2}\frac{z+2}{(z-1)(z-2)}(z-2) = 4$$

EXAMPLE-3:

Find the residue of $f(z) = \frac{e^{2z}}{z(z-1)}$ at Simple pole.

SOLUTION:

Here, $f(z) = \frac{e^{2z}}{z(z-1)}$

Pole is z (z-1) $\Rightarrow$ z_0=0 and z_0=1

Now,

Residue of f(z) at z_0=0

$$\therefore\ b_1 = Res(f(z),0) = \lim_{z\to 0}\frac{e^{2z}}{z(z-1)}z = \frac{e^0}{-1} = -1$$

Residue of f(z) at z_0=1

$$\therefore\ b_1 = Res(f(z),1) = \lim_{z\to 1}\frac{e^{2z}}{z(z-1)}(z-1) = e^2$$

EXAMPLE-4:

Find the residue of $f(z) = \frac{3z^2+2}{z(z-2)}$ at Simple pole.

SOLUTION:

Here, $f(z) = \frac{3z^2+2}{z(z-2)}$

Pole is z (z-2) $\Rightarrow$ z_0=0 and z_0=2

Now,

Residue of f(z) at z_0=0

$$\therefore\ b_1 = Res(f(z),0) = \lim_{z\to 0}\frac{3z^2+2}{z(z-2)}z = \frac{2}{-2} = -1$$

Residue of f(z) at z_0=2

$$\therefore\ b_1 = Res(f(z),2) = \lim_{z\to 2}\frac{3z^2+2}{z(z-2)}(z-2) = \frac{14}{2} = 7$$

EXAMPLE-5:

Find the residue of $f(z) = \frac{3z^2+1}{z(z-1)(z-2)}$ at Simple pole.

SOLUTION:

Here, $f(z) = \frac{3z^2+1}{z(z-1)(z-2)}$

Pole is z (z-1) (z-2) $\Rightarrow$ z_0=0, z_0=1 and z_0=2

Now,

Residue of f(z) at z_0=0

$$\therefore\ b_1 = Res(f(z),0) = \lim_{z\to 0}\frac{3z^2+1}{z(z-1)(z-2)}z = \frac{1}{(-1)(-2)} = \frac{1}{2}$$

Residue of f(z) at z_0=1

$$\therefore\ b_1 = Res(f(z),1) = \lim_{z\to 1}\frac{3z^2+1}{z(z-1)(z-2)}(z-1) = \frac{4}{(1)(-1)} = -4$$

Residue of f(z) at $z_0=2$

$$\therefore\ b_1 = Res(f(z),2) = \lim_{z\to 2}\frac{3z^2+1}{z(z-1)(z-2)}(z-2) = \frac{13}{(2)(1)} = \frac{13}{2}$$

EXAMPLE-6:

Find the residue of $f(z) = \frac{e^z}{z(z+1)}$ at Simple pole.

SOLUTION:

Here, $f(z) = \frac{e^z}{z(z+1)}$

Pole is z (z+1) $\Rightarrow$ $z_0=0$ and $z_0=-1$

Now,

Residue of f(z) at $z_0=0$

$$\therefore\ b_1 = Res(f(z),0) = \lim_{z\to 0}\frac{e^z}{z(z+1)}z = \frac{e^0}{(1)} = 1$$

Residue of f(z) at $z_0=-1$

$$\therefore\ b_1 = Res(f(z),-1) = \lim_{z\to -1}\frac{e^z}{z(z+1)}(z+1) = \frac{e^{-1}}{(-1)} = -e^{-1} = -\frac{1}{e}$$

EXAMPLE-7:

Find the residue of $f(z) = \frac{e^{3z}}{(z-1)(z^2+4)}$ at Simple pole.

SOLUTION:

Here, $f(z) = \frac{e^{3z}}{(z-1)(z^2+4)} = \frac{e^{3z}}{(z-1)(z+2i)(z-2i)}$

Pole is (z-1) (z+2i) (z-2i) $\Rightarrow$ z_0=1, z_0=-2i and z_0=2i

Now,

Residue of f(z) at z_0=1

$$\therefore\ b_1 = Res(f(z),1) = \lim_{z\to 1}\frac{e^{3z}}{(z-1)(z+2i)(z-2i)}(z-1) = \frac{e^3}{(1+2i)(1-2i)} = \frac{e^3}{1-4i^2} = \frac{e^3}{5}$$

Residue of f(z) at z_0=-2i

$$\therefore\ b_1 = Res(f(z),-2i) = \lim_{z\to -2i}\frac{e^{3z}}{(z-1)(z+2i)(z-2i)}(z+2i) = \frac{e^{-6i}}{(-2i-1)(-2i-2i)} = \frac{e^{-6i}}{(-2i-1)(-4i)} = \frac{e^{-6i}}{8i^2+4i} = \frac{e^{-6i}}{4(i-2)}$$

Residue of f(z) at z_0=2i

$$\therefore\ b_1 = Res(f(z),2i) = \lim_{z\to 2i}\frac{e^{3z}}{(z-1)(z+2i)(z-2i)}(z-2i) = \frac{e^{6i}}{(2i-1)(2i+2i)} = \frac{e^{6i}}{(2i-1)(4i)} = \frac{e^{6i}}{8i^2-4i} = \frac{e^{6i}}{-4(i+2)}$$

EXERCISE-A

Find the residue of following functions at Simple pole.

1) $f(z) = \frac{e^{2z}}{z(z^2+1)}$
2) $f(z) = \frac{z^3+2z}{(z-1)(z-2)}$
3) $f(z) = \frac{2z+1}{z(z-1)}$

Sr. No.	Question	Answer
1	If f(z) is an analytic everywhere in nbhd of z_0, except z_0, then the coefficient of $\frac{1}{z-z_0}$ in Laurent's expansion is called…….	**Residue of f(z)**
2	Give the formula to fond residue of f(z).	$\boldsymbol{b_1 = Res(f(z), z_0) = \frac{1}{2\pi i}\int_C f(z)dz}$
3	Write down the value of z_0 for the function $f(z) = \frac{e^z}{z(z+1)}$.	**z_0=0 & -1**

THEOREM:

If z_0 is the m^{th} order pole of complex function f(z) then prove that

$$Res(f(z), z_0) = \frac{\emptyset^{m-1}(z_0)}{(m-1)!}$$

Where,

$\emptyset(z) = (z - z_0)^m f(z)$

OR

Obtain the formula for finding the residue of f(z) at m^{th} order pole.

PROOF:

Let z_0 is m^{th} order pole.

By the definition of pole,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{(z - z_0)} + \frac{b_2}{(z - z_0)^2} + \; \ldots . . + \frac{b_m}{(z - z_0)^m}$$

$\therefore f(z)(z - z_0)^m$

$$= \sum_{n=0}^{\infty} a_n (z - z_0)^{n+m} + b_1 (z - z_0)^{m-1} + b_2 (z - z_0)^{m-2} + \; \ldots + b_m$$

Let $\emptyset(z) = (z - z_0)^m f(z)$

$$\therefore \; \emptyset(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{n+m} + b_1 (z - z_0)^{m-1} + b_2 (z - z_0)^{m-2} + \; \ldots + b_m$$

………………………… (1)

Now, we expand $\emptyset(z)$ by using Taylor's series

$$\emptyset(z) = \emptyset(z_0) + (z - z_0)\emptyset'(z_0) + \frac{(z - z_0)^2}{2!}\emptyset''(z_0) + \ldots + \frac{(z - z_0)^{m-1}}{(m-1)!}\emptyset^{m-1}(z_0)$$

……………………… (2)

Now,

We compare co-efficient of $(z-z_0)^{m-1}$ of equation (1) & (2), then we have

$$b_1 = \frac{\emptyset^{m-1}(z_0)}{(m-1)!}$$

Where,

$\emptyset(z) = (z - z_0)^m f(z)$

$\therefore$ Residue of f(z) at m^{th} order pole,

$Res(f(z), z_0) = b_1 = \frac{\emptyset^{m-1}(z_0)}{(m-1)!}$, Where, $\emptyset(z) = (z - z_0)^m f(z)$

EXAMPLE-8:

Find $Res\left(\frac{1-e^z}{z^4}, 0\right)$

SOLUTION:

Here, $f(z) = \frac{1-e^z}{z^4}$

Pole is $z^4 \Rightarrow z_0=0$ and $z_0=0$ is 4^{th} order pole.

Now,

$$b_1 = \frac{\emptyset^{m-1}(z_0)}{(m-1)!} = \frac{\emptyset^3(0)}{3!} \qquad \dots\dots\dots\dots\dots\dots (1)$$

We know that

$$\emptyset(z) = (z-z_0)^m f(z) = z^4\left(\frac{1-e^z}{z^4}\right) = 1-e^z$$

$$\emptyset'(z) = -e^z$$

$$\emptyset^2(z) = -e^z$$

$$\emptyset^3(z) = -e^z \Rightarrow \emptyset^3(0) = -e^0 = -1$$

By equation (1),

$$b_1 = \frac{\emptyset^3(0)}{3!} = \frac{-1}{3!} = -\frac{1}{6}$$

EXAMPLE-9:

Find $Res\left(\frac{ze^{iz}}{(z-3)^2}, 3\right)$

SOLUTION:

Here, $f(z) = \frac{ze^{iz}}{(z-3)^2}$

Pole is $(z-3)^2 \Rightarrow z_0=3$ and $z_0=3$ is 2^{nd} order pole.

Now,

$$b_1 = \frac{\emptyset^{m-1}(z_0)}{(m-1)!} = \frac{\emptyset'(3)}{1!} = \emptyset'(3) \qquad \dots\dots\dots\dots\dots\dots (1)$$

We know that

$$\emptyset(z) = (z-z_0)^m f(z) = (z-3)^2\left(\frac{ze^{iz}}{(z-3)^2}\right) = ze^{iz}$$

$$\emptyset'(z) = ze^{iz}i + e^{iz} = e^{iz}(iz+1)$$

$$\therefore \emptyset'(3) = e^{3i}(3i+1)$$

By equation (1),

$$b_1 = \emptyset'(3) = e^{3i}(3i+1)$$

EXAMPLE-10:

Find $Res\left(\frac{e^{2z}}{(z-1)^2}, 1\right)$

SOLUTION:

Here, $f(z) = \frac{e^{2z}}{(z-1)^2}$

Pole is $(z-1)^2 \Rightarrow z_0=1$ and $z_0=1$ is 2^{nd} order pole.

Now,

$$b_1 = \frac{\emptyset^{m-1}(z_0)}{(m-1)!} = \frac{\emptyset'(1)}{1!} = \emptyset'(1) \qquad \dots\dots\dots\dots\dots\dots (1)$$

We know that

$$\emptyset(z) = (z-z_0)^m f(z) = (z-1)^2\left(\frac{e^{2z}}{(z-1)^2}\right) = e^{2z}$$

$$\emptyset'(z) = 2e^{2z}$$

$$\therefore \emptyset'(1) = 2e^2$$

By equation (1),

$$b_1 = \emptyset'(1) = 2e^2$$

EXAMPLE-11:

Find $Res\left(\frac{e^{2z}}{z^2(z^2+1)}, z_0\right)$

SOLUTION:

Here, $f(z) = \frac{e^{2z}}{z^2(z^2+1)} = \frac{e^{2z}}{z^2(z-i)(z+i)}$

Pole is z^2 (z-i) (z+i) $\Rightarrow$ z_0=0, z_0=i, z_0=-i

For z_0=0 and z_0=0 is 2nd order pole.

Now,

$$b_1 = \frac{\emptyset^{m-1}(z_0)}{(m-1)!} = \frac{\emptyset'(0)}{1!} = \emptyset'(0) \qquad \dots\dots\dots\dots\dots\dots (1)$$

We know that

$$\emptyset(z) = (z - z_0)^m f(z) = z^2\left(\frac{e^{2z}}{z^2(z^2+1)}\right) = \frac{e^{2z}}{z^2+1}$$

$$\emptyset'(z) = \frac{(z^2+1)e^{2z}(2) - e^{2z}(2z)}{(z^2+1)^2} = \frac{2e^{2z}(z^2 - z + 1)}{(z^2+1)^2}$$

$$\therefore \emptyset'(0) = 2e^0(1) = 2$$

By equation (1),

$$b_1 = \emptyset'(0) = 2$$

Now,

Residue of f(z) at z_0=i

$$\therefore\ b_1 = Res(f(z), i) = \lim_{z\to i} \frac{e^{2z}}{z^2(z+i)(z-i)}(z - i) = \frac{e^{2i}}{i^2(i+i)} = \frac{e^{2i}}{(-1)(2i)} = \frac{ie^{2i}}{2}$$

Now,

Residue of f(z) at z_0=-i

$$\therefore\ b_1 = Res(f(z), -i) = \lim_{z\to -i} \frac{e^{2z}}{z^2(z+i)(z-i)}(z + i) = \frac{e^{-2i}}{(-i)^2(-i-i)} = \frac{e^{-2i}}{(-1)(-2i)} = \frac{-ie^{-2i}}{2}$$

Sr. No.	Question	Answer
1	Write down the formula for finding the residue of f(z) at m[th] order pole.	$\boldsymbol{Res(f(z), z_0) = \frac{\emptyset^{m-1}(z_0)}{(m-1)!}}$
2	What is the order of Pole z^4?	**4[th] order**

THEOREM:

State and prove Cauchy residue theorem.

STATEMENT:

If the complex function f(z) is an analytic everywhere inside and on the closed contour C, except the finite number of poles (singular points z_1, z_2, ………., z_n lying inside contour C) then,

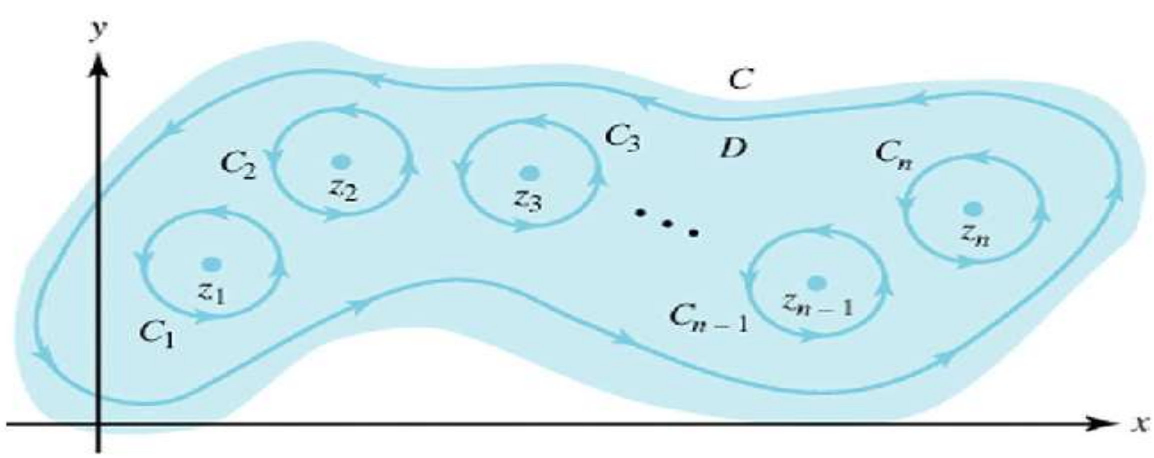
$$\int_C f(z)dz = 2\pi i[k_1 + k_2 + \dots\dots\dots + k_n]$$

OR

$$\int_C f(z)dz = 2\pi i \sum_{i=1}^{n} k_i, where, k_i = Res(f(z), z_i), i = 1,2, \dots.., n$$

PROOF:

- Let the complex function f(z) is an analytic everywhere inside and on the closed contour C, except the finite number of poles z_1, z_2, …….., z_n lying inside contour C.



- Now we draw a very small circles $C_1, C_2, \ldots\ldots, C_n$ with centre $z_1, z_2, \ldots\ldots, z_n$ respectively.
- In such a way boundaries of these circles and boundary of contour C are distinct.
- Thus, the complex function f(z) is analytic inside contour C and exterior region of the circles $C_1, C_2, \ldots\ldots, C_n$.

$\therefore$ By Cauchy fundamental theorem, line integral of f(z) is zero for this region.

$$\therefore \int_C f(z)dz - \int_{C_1} f(z)dz - \int_{C_2} f(z)dz - \ldots\ldots\ldots. - \int_{C_n} f(z)dz = 0$$

$$\therefore \int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \ldots\ldots\ldots. + \int_{C_n} f(z)dz \qquad \ldots\ldots.(i)$$

Let $k_i = Res(f(z), z_i),\ where\ i = 1, 2, \ldots\ldots\ldots., n$

Now,

$b_1 = Res(f(z), z) = k_1$

$b_1 = \frac{1}{2\pi i}\int_{C_1} f(z)dz$

$\therefore\ k_1 = \frac{1}{2\pi i}\int_{C_1} f(z)dz$

$$\int_{C_1} f(z)dz = 2\pi i \cdot k_1$$

Now,

$$k_2 = \frac{1}{2\pi i}\int_{C_2} f(z)dz$$

$$\therefore \int_{C_2} f(z)dz = 2\pi i \cdot k_2$$

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$$\int_{C_n} f(z)dz = 2\pi i \cdot k_n$$

Adding above term in equation (i),

$$\int_{C} f(z)dz = 2\pi i \cdot k_1 + 2\pi i \cdot k_2 + \ \dots\dots..+2\pi i \cdot k_n = 2\pi i[k_1 + k_2 + \ \dots\dots + k_n]$$

OR

$$\int_{C} f(z)dz = 2\pi i \sum_{i=1}^{n} k_i \ ,where\ k_i = Res(f(z), z_i), \qquad i = 1, 2, \dots\dots., n$$

EXAMPLE-12:

Evaluate

$$\int_{C} \frac{5z-2}{z(z-1)} dz\ ;\ C: |z| = 2$$

SOLUTION:

Here, $f(z) = \frac{5z-2}{z(z-1)}$

For poles z(z-1) $\Rightarrow$ $z_0=0$ and $z_0=1$

$Z_0=0$ & 1 ϵ C and $z_0=0$ & 1 is first order pole.

For $z_0=0$,

Residue of f(z) at $z_0=0$

$$k_1 = Res(f(z),0) = \lim_{z\to z_0}(z-z_0)f(z) = \lim_{z\to 0} z\left(\frac{5z-2}{z(z-1)}\right) = \frac{-2}{-1} = 2$$

For $z_0=1$,

Residue of f(z) at $z_0=1$

$$k_2 = Res(f(z),1) = \lim_{z\to z_0}(z-z_0)f(z) = \lim_{z\to 1}(z-1)\left(\frac{5z-2}{z(z-1)}\right) = 3$$

Now, By Cauchy residue theorem,

$$\int_C \frac{5z-2}{z(z-1)}dz = 2\pi i[k_1+k_2] = 2\pi i[2+3] = 10\pi i$$

EXAMPLE-13:

Evaluate

$$\int_C \frac{1-2z}{z(z-1)(z-2)}dz\ ;\ C:|z| = \frac{3}{2}$$

SOLUTION:

Here, $f(z) = \frac{1-2z}{z(z-1)(z-2)}$

For poles z(z-1)(z-2) $\Rightarrow$ z_0=0 , z_0=1 and z_0=2

Z_0=0 & 1 ϵ C but z_0=2 does not belongs to C and z_0=0 & 1 is first order pole.

For z_0=0,

Residue of f(z) at z_0=0

$$k_1 = Res(f(z),0) = \lim_{z\to z_0}(z-z_0)f(z) = \lim_{z\to 0} z\left(\frac{1-2z}{z(z-1)(z-2)}\right) = \frac{1}{(-1)(-2)}$$
$$= \frac{1}{2}$$

For z_0=1,

Residue of f(z) at z_0=1

$$k_2 = Res(f(z),1) = \lim_{z\to z_0}(z-z_0)f(z) = \lim_{z\to 1}(z-1)\left(\frac{1-2z}{z(z-1)(z-2)}\right)$$
$$= \frac{1-2}{1(1-2)} = 1$$

Now, By Cauchy residue theorem,

$$\int_C \frac{1-2z}{z(z-1)(z-2)}dz = 2\pi i[k_1+k_2] = 2\pi i\left[\frac{1}{2}+1\right] = 2\pi i\left(\frac{3}{2}\right) = 3\pi i$$

EXAMPLE-14:

Evaluate

$$\int_C z^{-3} e^{-z} dz , \qquad C: |z| = 1$$

SOLUTION:

Here, $f(z) = \frac{e^{-z}}{z^3}$

Poles $z^3 \Rightarrow z_0=0$

$Z_0=0 \in C$ and $z_0=0$ 3rd order pole.

$$k_1 = \frac{\emptyset^2(0)}{2!} \qquad \ldots\ldots\ldots\ldots..(i)$$

Now,

$$\emptyset(z) = e^{-z}$$

$$\emptyset'(z) = -e^{-z}$$

$$\emptyset^2(z) = e^{-z}$$

$$\therefore \ \emptyset^2(0) = 1$$

By equation (i),

$$k_1 = \frac{\emptyset^2(0)}{2!} = \frac{1}{2}$$

Now by Cauchy residue theorem,

$$\int_C z^{-3} e^{-z} dz = 2\pi i k_1 = 2\pi i \left(\frac{1}{2}\right) = \pi i$$

EXAMPLE-15:

Evaluate

$$\int_{|z|=3} \frac{z \cdot e^{\pi i z}}{z^2 + 2z + 5} dz$$

SOLUTION:

Here,

$$f(z) = \frac{z \cdot e^{\pi i z}}{z^2 + 2z + 5}$$

For pole $z^2 + 2z + 5$

$$\therefore \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4 - 4(1)(5)}}{2(1)} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

$\therefore \ z_0 = -1 + 2i \ and \ z_0 = -1 - 2i$,

$$z_0 = -1 + 2i = (-1,2) \in C \ \& \ z_0 = -1 - 2i = (-1,-2) \in C$$

$and \ z_0 = -1 \pm 2i$ Is first order pole.

$\therefore Residue \ of \ f(z) \ at \ z_0 = -1 + 2i$

$$k_1 = Res(f(z), -1 + 2i)$$

$$= \lim_{z \to -1+2i} (z + 1 - 2i) \cdot \frac{z \cdot e^{\pi i z}}{(z + 1 - 2i)(z + 1 + 2i)}$$

$$= \frac{(-1 + 2i)e^{\pi i(-1+2i)}}{(-1 + 2i + 1 + 2i)} = \frac{(-1 + 2i) \cdot e^{-\pi i} \cdot e^{-2\pi}}{4i}$$

$$= \frac{(-1 + 2i) \cdot e^{-2\pi}[\cos(-\pi) + i \sin(-\pi)]}{4i} = \frac{(-1 + 2i)e^{-2\pi}(-1)}{4i}$$

$$= \frac{(1 - 2i)e^{-2\pi}}{4i}$$

Now,

$\therefore$ $Residue\ of\ f(z)\ at\ z_0 = -1 - 2i$

$k_2 = Res(f(z), -1 - 2i)$

$$= \lim_{z \to -1-2i} (z + 1 + 2i) \cdot \frac{z \cdot e^{\pi i z}}{(z + 1 - 2i)(z + 1 + 2i)}$$
$$= \frac{(-1 - 2i)e^{\pi i(-1-2i)}}{(-1 - 2i + 1 - 2i)} = \frac{(-1 - 2i) \cdot e^{-\pi i} \cdot e^{2\pi}}{-4i}$$
$$= \frac{(-1 - 2i)e^{2\pi}(-1)}{-4i} = -\frac{(1 + 2i)e^{2\pi}}{4i}$$

By Cauchy residue theorem,

$$\int_{|z|=3} \frac{z \cdot e^{\pi i z}}{z^2 + 2z + 5} dz = 2\pi i[k_1 + k_2] = 2\pi i \left[\frac{(1 - 2i)e^{-2\pi}}{4i} - \frac{(1 + 2i)e^{2\pi}}{4i}\right]$$
$$= \frac{2\pi i}{4i}[(1 - 2i)e^{-2\pi} - (1 + 2i)e^{2\pi}]$$
$$= \frac{\pi}{2}[(1 - 2i)e^{-2\pi} - (1 + 2i)e^{2\pi}]$$

EXAMPLE-16:

Evaluate

$$\int_{|z|=3} \frac{e^{tz}}{z^2(z^2 + 2z + 2)} dz$$

SOLUTION:

Here,

$$f(z) = \frac{e^{tz}}{z^2(z^2 + 2z + 2)} dz$$

Poles $z^2 \Rightarrow z_0=0$

For pole z^2+2z+2

$$\therefore \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-2 \pm \sqrt{4-4(1)(2)}}{2(1)} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$z_0 = -1 + i \text{ and } z_0 = -1 - i$

Now,

$Z_0=0$ is second order pole.

$$k_1 = \frac{\emptyset^{m-1}(z_0)}{(m-1)!} = \emptyset'(0) \qquad \dots\dots\dots\dots\dots\dots\dots.(i)$$

$$\emptyset(z) = (z-z_0)^m f(z) = z^2 \cdot \frac{e^{tz}}{z^2(z^2+2z+2)} = \frac{e^{tz}}{z^2+2z+2}$$

$$\therefore \ \emptyset'(z) = \frac{[(z^2+2z+2)\cdot e^{tz}\cdot t - e^{tz}(2z+2)]}{(z^2+2z+2)^2}$$

$$\therefore \ \emptyset'(0) = \frac{[(2)(t)-1(2)]}{4} = \frac{2t-2}{4} = \frac{t-1}{2}$$

By equation (i),

$$k_1 = \emptyset'(0) = \frac{t-1}{2}$$

$$z_0 = -1 \pm i \text{ is } 1^{st} \text{ order pole}$$

For $z_0=-1+i$

Residue of f(z) at $z_0=-1+i$

$$k_2 = \lim_{z \to z_0} (z - z_0) f(z)$$

$$= \lim_{z \to -1+i} (z + 1 - i) \cdot \frac{e^{tz}}{z^2 (z + 1 - i)(z + 1 + i)}$$

$$= \frac{e^{t(-1+i)}}{(-1 + i)^2 (-1 + i + 1 + i)} = \frac{e^{-t} \cdot e^{it}}{(1 - 2i - 1)(2i)} = \frac{e^{-t} \cdot e^{it}}{4}$$

For z_0=-1-i

Residue of f(z) at z_0=-1-i

$$k_3 = \lim_{z \to z_0} (z - z_0) f(z)$$

$$= \lim_{z \to -1-i} (z + 1 + i) \cdot \frac{e^{tz}}{z^2 (z + 1 - i)(z + 1 + i)}$$

$$= \frac{e^{t(-1-i)}}{(-1 - i)^2 (-1 - i + 1 - i)} = \frac{e^{-t} \cdot e^{-it}}{(1 + 2i - 1)(-2i)} = \frac{e^{-t} \cdot e^{-it}}{4}$$

By Cauchy residue theorem,

$$\int_{|z|=3} \frac{e^{tz}}{z^2 (z^2 + 2z + 2)} dz = 2\pi i [k_1 + k_2 + k_3]$$

$$= 2\pi i \left[\frac{t - 1}{2} + \frac{e^{-t} \cdot e^{it}}{4} + \frac{e^{-t} \cdot e^{-it}}{4} \right]$$

$$= 2\pi i \left[\frac{t - 1}{2} + \frac{e^{-t}}{2} \left(\frac{e^{it} + e^{-it}}{2} \right) \right] = 2\pi i \left[\frac{t - 1}{2} + \frac{e^{-t}}{2} \cdot \cos t \right]$$

$$= \pi i [t - 1 + e^{-t} cost]$$

EXERCISE-B

Find the residue of following functions:

1) $\int_C \frac{3z^2 + 2}{(z - 1)(z^2 + 9)} dz, \ where \ C: |z| = 2$
2) $\int_C \frac{2z + 3}{z(z - 1)} dz, \ where \ C: |z| = 2$

3) $\int_C \frac{e^z}{z(z-1)^2} dz, \ \ where\ C: |z| = 2$

4) $\int_C e^{-z} z^{-2} dz, \ \ where\ C: |z| = 1$

5) $\int_C \frac{z^2+2z}{(z+1)^2(z^2+4)} dz, \ where\ C: |z| = 3$

Sr. No.	Question	Answer
1	What is the value of singular point for pole $z^2 + 2z + 2$?	$-\mathbf{1} \pm \boldsymbol{i}$
2	First order pole is also known as	**Simple pole**

♣ Definite integral of trigonometric functions:

To evaluate

$$\int_0^{2\pi} F[\cos\theta, \sin\theta]\, d\theta$$

Put $z = e^{i\theta}$

$$dz = ie^{i\theta} d\theta \ \Rightarrow d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+z^{-1}}{2} = \frac{z+{}^{1}/_{z}}{2} = \frac{z^2+1}{2z}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i} = \frac{z - {}^{1}/_{z}}{2i} = \frac{z^2 - 1}{2iz}$$

Thus,

$$\int_0^{2\pi} F[\cos\theta, \sin\theta]\, d\theta = \int_C f(z)dz, \qquad C: |z| = 1$$

EXAMPLE-17:

Evaluate

$$\int_0^{2\pi} \frac{1}{1 - 2a\cos\theta + a^2} d\theta; \ \ a^2 < 1$$

SOLUTION:

$$z = e^{i\theta}$$

$$dz = e^{i\theta} \cdot i\, d\theta \ \ \Rightarrow \ d\theta = \frac{dz}{iz}$$

$$z = \cos\theta + i\sin\theta$$

$$\frac{1}{z} = \cos\theta - i\sin\theta$$

$$\therefore z + \frac{1}{z} = 2\cos\theta \ \ \Rightarrow \ \cos\theta = \frac{z^2+1}{2z}$$

Now,

$$\int_C \frac{1}{1 - 2a\left(\frac{z^2+1}{2z}\right) + a^2} \cdot \frac{dz}{iz}$$

$$= \frac{1}{i}\int_C \frac{1}{1 - \frac{az^2 - a}{z} + a^2} \cdot \frac{dz}{z} = \frac{1}{i}\int_C \frac{1}{z - az^2 - a + az^2} dz$$

$$= \frac{1}{i}\int_C \frac{1}{(z-a)(1-az)} dz$$

Here,

$$f(z) = \frac{1}{(z-a)(1-az)}, \qquad z_0 = a \in C \; and \; z_0 = \frac{1}{a} \; does\ not\ belongs\ to\ C$$

For z_0=a is first order pole.

Residue of f(z) at z_0=a

$$k_1 = \lim_{z\to z_0} f(z)(z-z_0) = \lim_{z\to a} \frac{1}{(z-a)(1-az)} \cdot (z-a) = \frac{1}{1-a^2}$$

Therefore, By Cauchy residue theorem,

$$\int_0^{2\pi} \frac{1}{1-2a\cos\theta + a^2} d\theta = 2\pi i(k_1) \cdot \frac{1}{i} = \frac{2\pi i}{i}\left(\frac{1}{1-a^2}\right) = \frac{2\pi}{1-a^2}$$

EXAMPLE-18:

Evaluate

$$\int_0^{\pi} \frac{1}{(2+\cos\theta)^2} d\theta$$

SOLUTION:

We know that

$$\cos\theta = \frac{z^2+1}{2z}, \qquad d\theta = \frac{dz}{iz}$$

Now,

$$\int_C \frac{1}{\left(2+\left(\frac{z^2+1}{2z}\right)\right)^2}\cdot\frac{dz}{iz}=\frac{1}{i}\int_C \frac{1}{\left(\frac{4z+z^2+1}{2z}\right)^2}\cdot\frac{dz}{z}=\frac{1}{i}\int_C \frac{4z}{(z^2+4z+1)^2}dz$$

$$z_0=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-4\pm\sqrt{16-4(1)(1)}}{2}=\frac{-4\pm2\sqrt{3}}{2}=-2\pm\sqrt{3}$$

$z_0=-2+\sqrt{3}\in C$ *and* $z_0=-2-\sqrt{3}$ *does not belongs to* C

For $z_0=-2+\sqrt{3}$ is second order pole.

$$k_1=\frac{\emptyset^{m-1}(z_0)}{(m-1)!}=\emptyset'\left(-2+\sqrt{3}\right) \qquad \dots\dots\dots\dots\dots\dots(i)$$

Now,

$$\emptyset(z)=(z-z_0)^m\cdot f(z)=(z+2-\sqrt{3})^2\cdot\frac{4z}{(z+2-\sqrt{3})^2(z+2+\sqrt{3})^2}$$

$$=\frac{4z}{(z+2+\sqrt{3})^2}$$

$$\emptyset'(z)=\frac{\left[(z+2+\sqrt{3})^2\cdot4-4z(2(z+2+\sqrt{3})\cdot1)\right]}{(z+2+\sqrt{3})^4}$$

$$\therefore\ \emptyset'\left(-2+\sqrt{3}\right)=\frac{4\left[(-2+\sqrt{3}+2+\sqrt{3})^2-2\left(-2+\sqrt{3}\right)\left(-2+\sqrt{3}+2+\sqrt{3}\right)\right]}{(-2+\sqrt{3}+2+\sqrt{3})^4}$$

$$=\frac{4\left[12-2\left(-2+\sqrt{3}\right)(2\sqrt{3})\right]}{144}=\frac{16}{144}\left[3-(-2\sqrt{3}+3)\right]$$

$$=\frac{1}{9}\left[3+2\sqrt{3}-3\right]=\frac{2\sqrt{3}}{9}$$

By equation (i), $k_1=\frac{2\sqrt{3}}{9}$

$$\therefore\int_0^\pi\frac{1}{(2+\cos\theta)^2}d\theta=\pi i(k_1)\cdot\frac{1}{i}=\frac{\pi i}{i}\left(\frac{2\sqrt{3}}{9}\right)=\frac{2\pi}{3\sqrt{3}}$$

EXAMPLE-19:

Prove that

$$\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta = \frac{\pi}{6}$$

SOLUTION:

We know that

$$\cos\theta = \frac{z^2+1}{2z}, \qquad d\theta = \frac{dz}{iz}$$

$$\cos 2\theta = \frac{z^2+z^{-2}}{2} = \frac{z^2 + {}^1/_{z^2}}{2} = \frac{z^4+1}{2z^2}$$

Now,

$$\int_C \frac{\left(\frac{z^4+1}{2z^2}\right)}{5+4\left(\frac{z^2+1}{2z}\right)} \cdot \frac{dz}{iz} = \frac{1}{i}\int_C \frac{z^4 + {}^1/_{2z^2}}{\left(\frac{10z+4z^2+4}{2z}\right)} \cdot dz$$

$$= \frac{1}{i}\int_C \frac{z^4+1}{z^2(4z^2+10z+4)} dz = \frac{1}{i}\int_C \frac{z^4+1}{z^2(z+2)(4z+2)} dz$$

$$z_0 = 0, z_0 = -2, z_0 = -\frac{1}{2}$$

$$z_0 = 0, -\frac{1}{2} \in C \ and\ z_0 = -2\ does\ not\ belongs\ to\ C$$

For $z_0 = 0$ is second order pole.

$$k_1 = \frac{\emptyset^{m-1}(z_0)}{(m-1)!} = \emptyset'(0) \qquad \dots\dots\dots\dots\dots\dots\dots..(i)$$

Now,

$$\emptyset(z) = (z - z_0)^m f(z) = z^2 \cdot \frac{z^4+1}{z^2(z+2)(4z+2)} = \frac{z^4+1}{4z^2+10z+4}$$

$$\emptyset'(z) = \frac{[(4z^2+10z+4)\cdot 4z^3 - (z^4+1)(8z+10)]}{(4z^2+10z+4)^2}$$

$$\therefore\ \emptyset'(0) = \frac{[0-(1)(10)]}{16} = -\frac{10}{16} = -\frac{5}{8}$$

By equation (i), $k_1 = -\frac{5}{8}$

For $z_0 = -\frac{1}{2}$ is first order pole.

Residue of f(z) at $z_0 = -\frac{1}{2}$

$$k_2 = \lim_{z\to z_0}(z-z_0)f(z)$$

$$= \lim_{z\to -\frac{1}{2}}\left(z+\frac{1}{2}\right)\cdot\frac{z^4+1}{z^2(z+2)(4z+2)}$$

$$= \lim_{z\to -\frac{1}{2}}\left(z+\frac{1}{2}\right)\cdot\frac{z^4+1}{z^2(z+2)4(z+\frac{1}{2})} = \left[\frac{{}^1/_{16}+1}{{}^1/_4\,({}^3/_2)\cdot 4}\right] = \frac{17}{24}$$

By Cauchy Residue theorem,

$$\int_0^{2\pi}\frac{\cos 2\theta}{5+4\cos\theta}\,d\theta = 2\pi i(k_1+k_2)\cdot\frac{1}{i} = \frac{2\pi i}{i}\left[-\frac{5}{8}+\frac{17}{24}\right] = 2\pi\left[\frac{2}{24}\right] = \frac{\pi}{6}$$

EXAMPLE-20:

Show that

$$\int_0^{2\pi} \frac{sin^2\theta}{a + b\cos\theta} d\theta = \frac{2\pi}{b^2}\left(a - b\sqrt{a^2 - b^2}\right); \quad (a > b)$$

SOLUTION:

Let C is unit circle.

We know that,

$z = e^{i\theta}$

$dz = e^{i\theta} \cdot id\theta \;\Rightarrow d\theta = \dfrac{dz}{iz}$

And

$\cos\theta = \dfrac{z^2 + 1}{2z}, \;\; \sin\theta = \dfrac{z^2 - 1}{2iz}$

Now,

$$\int_C \frac{\left(\frac{z^2 - 1}{2iz}\right)^2}{a + b\left(\frac{z^2 + 1}{2z}\right)} \cdot \frac{dz}{iz}$$

$$= \frac{1}{i}\int_C \frac{(z^2 - 1)^2 / -4z^2}{\frac{2az + bz^2 + b}{2z}} \cdot \frac{dz}{z} = -\frac{1}{2i}\int_C \frac{(z^2 - 1)^2}{z^2(bz^2 + 2az + b)} dz$$

For pole $z^2 = 0 \;\&\; bz^2 + 2az + b = 0$

$$z_0 = 0 \ \& \ z_0 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2a \pm \sqrt{4a^2 - 4b^2}}{2b} = \frac{-a \pm \sqrt{a^2 - b^2}}{b}$$

$$\therefore \ z_0 = 0 \ \& \ z_0 = \frac{-a + \sqrt{a^2 - b^2}}{b}$$

$$\in C \ but \ \frac{-a - \sqrt{a^2 - b^2}}{b} \ does \ not \ belongs \ to \ C; (\because a > b)$$

For $z_0 = 0$ is 2nd order pole.

$$k_1 = \frac{\emptyset^{m-1}(z_0)}{(m-1)!} = \emptyset'(0) \qquad \dots \dots \dots \dots \dots \dots \dots \dots (i)$$

Now,

$$\emptyset(z) = (z - z_0)^m f(z) = z^2 \cdot \frac{(z^2 - 1)^2}{z^2(bz^2 + 2az + b)} = \frac{(z^2 - 1)^2}{(bz^2 + 2az + b)}$$

$$\therefore \ \emptyset'(z) = \frac{[(bz^2 + 2az + b) \cdot 2(z^2 - 1) \cdot 2z - (z^2 - 1)^2(2bz + 2a)]}{(bz^2 + 2az + b)^2}$$

$$\therefore \ \emptyset'(0) = \frac{0 - (-1)^2(2a)}{b^2} = -\frac{2a}{b^2}$$

By equation (i), $k_1 = -\frac{2a}{b^2}$

For $z_0 = \frac{-a + \sqrt{a^2 - b^2}}{b}$ is first order pole.

Residue of *f(z)* at z_0,

$$k_2 = \lim_{z \to \frac{-a+\sqrt{a^2-b^2}}{b}} \frac{\left(\frac{z+a-\sqrt{a^2-b^2}}{b}\right)\cdot (z^2-1)^2}{z^2 \cdot \left(\frac{z+a-\sqrt{a^2-b^2}}{b}\right)\left(\frac{z+a+\sqrt{a^2-b^2}}{b}\right)}$$

$$= \frac{\left[\left(\frac{-a+\sqrt{a^2-b^2}}{b}\right)^2 - 1\right]^2}{\left(\frac{-a+\sqrt{a^2-b^2}}{b}\right)^2\left(\frac{-a+\sqrt{a^2-b^2}}{b} + \frac{a+\sqrt{a^2-b^2}}{b}\right)}$$

$$= \frac{\frac{\left(a^2-2a\sqrt{a^2-b^2}+a^2-b^2-b^2\right)^2}{b^4}}{\frac{\left(-a+\sqrt{a^2-b^2}\right)^2}{b^2}\cdot\frac{2\sqrt{a^2-b^2}}{b}}$$

$$= \frac{\frac{\left(2a^2-2a\sqrt{a^2-b^2}-2b^2\right)^2}{b^4}}{\frac{2\left(-a+\sqrt{a^2-b^2}\right)^2\left(\sqrt{a^2-b^2}\right)}{b^3}}$$

$$= \frac{4}{2b}\left[\frac{\left(a^2-a\sqrt{a^2-b^2}-b^2\right)^2}{\left(\sqrt{a^2-b^2}\right)\left(-a+\sqrt{a^2-b^2}\right)^2}\right]$$

$$= \frac{2}{b}\left\{\frac{\left[\left(\sqrt{a^2-b^2}\right)\left(\sqrt{a^2-b^2}-a\right)\right]^2}{\left(\sqrt{a^2-b^2}\right)\left(\sqrt{a^2-b^2}-a\right)^2}\right\}$$

$$= \frac{2}{b}\left[\frac{\left(\sqrt{a^2-b^2}\right)^2\left(\sqrt{a^2-b^2}-a\right)^2}{\left(\sqrt{a^2-b^2}\right)\left(\sqrt{a^2-b^2}-a\right)^2}\right] = \frac{2}{b}\left(\sqrt{a^2-b^2}\right)$$

By Cauchy Residue theorem,

$$\therefore \frac{-1}{2i}\int_C \frac{(z^2-1)^2}{z^2(bz^2+2az+b)}dz = \frac{-1}{2i}\cdot 2\pi i[k_1+k_2]$$

$$= -\pi\left[\frac{-2a}{b^2} + \frac{2}{b}\left(\sqrt{a^2-b^2}\right)\right] = \frac{2\pi}{b^2}\left(a - b\sqrt{a^2-b^2}\right)$$

EXAMPLE-21:

Prove that

$$\int_0^{2\pi} \frac{\cos 2\theta}{1-2P\cos\theta+P^2}d\theta = \frac{2\pi P^2}{1-P^2}; \qquad (0<P<1)$$

SOLUTION:

Let C is unit circle.

We know that,

$$d\theta = \frac{dz}{iz}, \qquad \cos\theta = \frac{z^2+1}{2z}, \quad \cos 2\theta = \frac{z^4+1}{2z^2}$$

Now,

$$\int_C \frac{\left(\frac{z^4+1}{2z^2}\right)}{1-2P\left(\frac{z^2+1}{2z}\right)+P^2}\cdot\frac{dz}{iz} = \frac{1}{2i}\int_C \frac{\left(\frac{z^4+1}{z^3}\right)}{\frac{2z-2Pz^2-2P+2P^2z}{2z}}dz$$

$$= \frac{1}{2i}\int_C \frac{z^4+1}{z^2(z-Pz^2-P+P^2z)}dz = \frac{1}{2i}\int_C \frac{z^4+1}{z^2(z-P)(1-Pz)}dz$$

For pole $z^2(z-P)(1-Pz) \Rightarrow z_0 = 0, P, \frac{1}{P}$

$$z_0 = 0 \; and \; z_0 = P \in C \;\&\; z_0 = \frac{1}{P} \; does \; not \; belongs \; to \; C.$$

For z_0=0 is 2nd order pole.

$$k_1 = \frac{\emptyset^{m-1}(z_0)}{(m-1)!} = \emptyset'(0) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(i)$$

Now,

$$\emptyset(z) = (z-z_0)^m f(z) = z^2 \cdot \frac{z^4+1}{z^2(z-P)(1-Pz)} = \frac{z^4+1}{(z-Pz^2-P+P^2z)}$$

$$\therefore\ \emptyset'(z) = \frac{[(z-Pz^2-P+P^2z)(4z^3)-(z^4+1)(1-2Pz+P^2)]}{(z-Pz^2-P+P^2z)^2}$$

$$\therefore\ \emptyset'(0) = \frac{[0-(1)(1+P^2)]}{P^2} = \frac{-1-P^2}{P^2}$$

By equation (i), $k_1 = \frac{-1-P^2}{P^2}$

Z_0=P is first order pole.

Residue of f(z) at z_0=P,

$$k_2 = \lim_{z\to z_0}(z-z_0)f(z) = \lim_{z\to P}(z-P)\cdot\frac{z^4+1}{z^2(z-P)(1-Pz)} = \frac{P^4+1}{P^2(1-P^2)}$$

By Cauchy residue theorem,

$$\int_0^{2\pi}\frac{\cos 2\theta}{1-2P\cos\theta+P^2}d\theta = 2\pi i[k_1+k_2]\cdot\frac{1}{2i} = \frac{2\pi i}{2i}\left[\frac{-1-P^2}{P^2}+\frac{P^4+1}{P^2(1-P^2)}\right]$$

$$= \pi\left[\frac{(-1-P^2)(1-P^2)+P^4+1}{P^2(1-P^2)}\right]$$

$$= \pi\left[\frac{-1+P^2-P^2+P^4+P^4+1}{P^2(1-P^2)}\right] = \frac{\pi\cdot 2P^4}{P^2(1-P^2)} = \frac{2\pi P^2}{1-P^2}$$

EXAMPLE-22:

Prove that

$$\int_0^{\pi} \frac{a}{a^2 + sin^2\theta} d\theta = \frac{\pi}{\sqrt{1+a^2}}; \qquad (a > 1)$$

SOLUTION:

We know that,

$$\sin\theta = \frac{z^2-1}{2iz}, d\theta = \frac{dz}{iz}$$

Now,

$$\int_C \frac{a}{a^2 + \left(\frac{z^2-1}{2iz}\right)^2} \cdot \frac{dz}{iz}$$

$$= \frac{a}{i} \int_C \frac{1}{a^2 - \frac{(z^2-1)^2}{4z^2}} \cdot \frac{dz}{z}$$

$$= \frac{a}{i} \int_C \frac{4z}{4a^2z^2 - (z^2-1)^2} dz$$

$$= \frac{4a}{i} \int_C \frac{z}{(2az - z^2 + 1)(2az + z^2 - 1)} dz$$

$$= -\frac{4a}{i} \int_C \frac{z}{(z^2 - 2az - 1)(z^2 + 2az - 1)} dz$$

For pole $(z^2$-2az-1)$(z^2$+2az-1)

$$\therefore \; z_0 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2a \pm \sqrt{4a^2 + 4}}{2} = a \pm \sqrt{a^2+1}$$

$$\therefore \; z_0 = a + \sqrt{a^2+1} \; and \; z_0 = a - \sqrt{a^2+1}$$

Now,

$$z_0 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2a \pm \sqrt{4a^2 + 4}}{2} = -a \pm \sqrt{a^2 + 1}$$

$$\therefore\ z_0 = a - \sqrt{a^2 + 1}\ \&\ z_0 = -a + \sqrt{a^2 + 1} \in C \text{ and } z_0 = -a - \sqrt{a^2 + 1}\ \&\ z_0 = a + \sqrt{a^2 + 1} \text{ does not belongs to } C$$

For $z_0 = a - \sqrt{a^2 + 1}$ is first order pole.

Residue of f(z) at $z_0 = a - \sqrt{a^2 + 1}$

$$k_1 = \lim_{z \to a - \sqrt{a^2+1}} (z - a + \sqrt{a^2 + 1}) \cdot \frac{z}{\left(z - a + \sqrt{a^2 + 1}\right)\left(z + a - \sqrt{a^2 + 1}\right)\left(z - a - \sqrt{a^2 + 1}\right)\left(z + a + \sqrt{a^2 + 1}\right)}$$

$$= \frac{a - \sqrt{a^2 + 1}}{\left(a - \sqrt{a^2 + 1} - a - \sqrt{a^2 + 1}\right)\left(a - \sqrt{a^2 + 1} + a - \sqrt{a^2 + 1}\right)\left(a - \sqrt{a^2 + 1} + a + \sqrt{a^2 + 1}\right)}$$

$$= \frac{a - \sqrt{a^2 + 1}}{\left(-2\sqrt{a^2 + 1}\right)\left(2a - 2\sqrt{a^2 + 1}\right)(2a)} = -\frac{1}{8a\sqrt{a^2 + 1}}$$

For $z_0 = -a + \sqrt{a^2 + 1}$ is first order pole.

$$k_2 = \lim_{z \to -a + \sqrt{a^2+1}} \left(z + a - \sqrt{a^2 + 1}\right) \cdot \frac{z}{\left(z + a - \sqrt{a^2 + 1}\right)\left(z + a + \sqrt{a^2 + 1}\right)\left(z - a + \sqrt{a^2 + 1}\right)\left(z - a - \sqrt{a^2 + 1}\right)}$$

$$= \frac{-a + \sqrt{a^2 + 1}}{\left(2\sqrt{a^2 + 1}\right)\left(-2a + 2\sqrt{a^2 + 1}\right)(-2a)} = -\frac{1}{8a\sqrt{a^2 + 1}}$$

By Cauchy residue theorem, we get

$$\int_0^{\pi} \frac{a}{a^2+sin^2\theta}d\theta = -\frac{4a}{i}\cdot \pi i[k_1+k_2] = -4a\pi\left[-\frac{1}{8a\sqrt{a^2+1}}-\frac{1}{8a\sqrt{a^2+1}}\right]$$

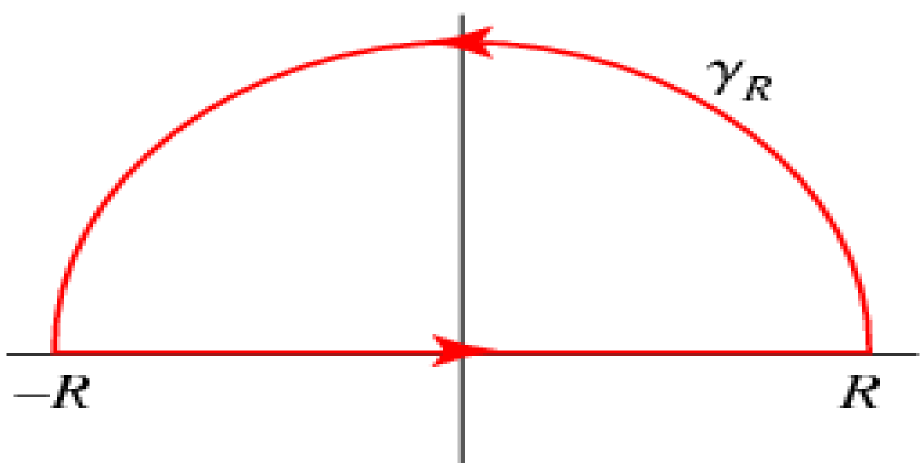
$$= \frac{8a\pi}{8a\sqrt{a^2+1}} = \frac{\pi}{\sqrt{a^2+1}}$$

EXERCISE-C

Find the residue of following functions:

1) $\int_0^{2\pi}\frac{d\theta}{\cos\theta+2} = \frac{2\pi}{\sqrt{3}}$
2) $\int_0^{2\pi}\frac{d\theta}{5+4\cos\theta} = \frac{2\pi}{3}$
3) $\int_0^{2\pi}\frac{d\theta}{1+a\cos\theta} = \frac{2\pi}{\sqrt{1-a^2}};\ -1<a<1$
4) $\int_0^{2\pi}\frac{\cos 3\theta\, d\theta}{5-4\cos\theta} = \frac{\pi}{12}$

♣ Evaluation of improper real integrals:



$$\int_{-\infty}^{\infty} F(x)dx = \lim_{R\to\infty} \left[\int_{-R}^{R} f(x)dx + \int_{C_R} f(x)dx\right]$$

$$where\ C_R\ is\ upper\ half\ of\ circle\ C: |z| = R$$

Now,

When R→∞ then $\int_{C_R} f(x)dx = 0$

$$\therefore \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx = 2\pi i(k_1 + k_2 + \dots\dots\dots\dots + k_n)$$

Where k_1, k_2, ……….., k_n are residue of f(x) at poles.

EXAMPLE-23:

Evaluate:

$$\int_{-\infty}^{\infty} \frac{dx}{(x+1)(x^2+2)}$$

SOLUTION:

Let

$$\int_{-\infty}^{\infty} \frac{dx}{(x+1)(x^2+2)} = \int_{-\infty}^{\infty} \frac{dz}{(z+1)(z^2+2)}$$

For pole $(z+1)(z^2+2) \Rightarrow z_0 = -1, z_0 = \pm\sqrt{2}$

$z_0 = -1, \sqrt{2}i \in C\ and\ z_0 = -\sqrt{2}i\ does\ not\ belongs\ to\ C$

For z_0=-1 is first order pole.

$$k_1 = \lim_{z \to -1} (z+1) \cdot \frac{1}{(z+1)(z^2+2)} = \frac{1}{3}$$

For $z_0 = \sqrt{2}i$ is first order pole.

$$k_2 = \lim_{z \to \sqrt{2}i} \left(z - \sqrt{2}i\right) \cdot \frac{1}{(z+1)\left(z-\sqrt{2}i\right)\left(z+\sqrt{2}i\right)} = \frac{1}{\left(\sqrt{2}i+1\right)\left(2\sqrt{2}i\right)}$$

$$= \frac{1}{-4+2\sqrt{2}i} = \frac{1}{2\sqrt{2}i-4} \times \frac{2\sqrt{2}i+4}{2\sqrt{2}i+4} = \frac{2\sqrt{2}i+4}{-8-16} = \frac{\sqrt{2}i+2}{(-12)}$$

By Cauchy residue theorem,

$$\int_{-\infty}^{\infty} \frac{dz}{(z+1)(z^2+2)} = 2\pi i[k_1+k_2] = 2\pi i\left[\frac{1}{3} + \frac{\sqrt{2}i+2}{(-12)}\right] = \frac{2\pi i}{3}\left[1 - \frac{(\sqrt{2}i+2)}{4}\right]$$

$$= \frac{2\pi i}{3}\left[\frac{4-\sqrt{2}i-2}{4}\right] = \frac{\pi i}{6}(2-\sqrt{2}i)$$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{(x+1)(x^2+2)} = \frac{\pi i}{6}(2-\sqrt{2}i)$$

EXAMPLE-24:

Evaluate:

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx$$

SOLUTION:

Let

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}dx = \int_{-\infty}^{\infty} \frac{z^2}{(z^2+1)(z^2+4)}dz$$

For pole $(z^2+1)(z^2+4) \Rightarrow z_0 = \pm i,\ z_0 = \pm 2i$

$z_0 = i \text{ and } 2i \in C \text{ and } z_0 = -i \text{ and } -2i \text{ does not belongs to } C$

For $z_0 = i$ is first order pole.

$$k_1 = \lim_{z\to i}(z-i)\cdot\frac{z^2}{(z-i)(z+i)(z+2i)(z-2i)} = \frac{i^2}{(2i)(3i)(-i)} = \frac{-1}{-6(-i)}$$
$$= -\frac{1}{6i}$$

For $z_0 = 2i$ is first order pole.

$$k_2 = \lim_{z\to 2i}(z-2i)\cdot\frac{z^2}{(z-i)(z+i)(z+2i)(z-2i)} = \frac{4i^2}{(3i)(i)(4i)} = \frac{-4}{-12i} = \frac{1}{3i}$$

By Cauchy residue theorem,

$$\int_{-\infty}^{\infty} \frac{z^2}{(z^2+1)(z^2+4)}dz = 2\pi i[k_1+k_2] = 2\pi i\left[-\frac{1}{6i}+\frac{1}{3i}\right] = \frac{2\pi i}{3i}\left[-\frac{1}{2}+1\right]$$
$$= \frac{2\pi}{3}\left[\frac{1}{2}\right] = \frac{\pi}{3}$$

$$\therefore \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}dx = \frac{\pi}{3}$$

EXAMPLE-25:

Evaluate:

$$\int_{-\infty}^{\infty} \frac{x^2+x+3}{x^4+5x^2+4}dx$$

SOLUTION:

Let

$$\int_{-\infty}^{\infty} \frac{x^2+x+3}{x^4+5x^2+4}dx = \int_{-\infty}^{\infty} \frac{z^2+z+3}{z^4+5z^2+4}dz$$

For pole

$$z^4+5z^2+4=0 \Rightarrow z^4+z^2+4z^2+4=0 \Rightarrow z^2(z^2+1)+4(z^2+1)=0$$

$$(z^2+1)(z^2+4)=0$$

$$\therefore\ z_0=\pm i \text{ and } z_0=\pm 2i$$

$$z_0 = i \,\&\, 2i \in C \text{ and } z_0=-i \,\&-2i \text{ does not belongs to } C$$

For $z_0 = i$ is first order pole.

$$k_1 = \lim_{z\to i}(z-i)\cdot\frac{z^2+z+3}{(z-i)(z+i)(z+2i)(z-2i)} = \frac{i^2+i+3}{(2i)(3i)(-i)} = \frac{2+i}{6i}$$

For $z_0 = 2i$ is first order pole.

$$k_2 = \lim_{z\to 2i}(z-2i)\cdot\frac{z^2+z+3}{(z-i)(z+i)(z+2i)(z-2i)} = \frac{4i^2+2i+3}{(i)(3i)(4i)} = \frac{2i-1}{-12i}$$

$$= \frac{1-2i}{12i}$$

By Cauchy residue theorem,

$$\int_{-\infty}^{\infty} \frac{z^2+z+3}{z^4+5z^2+4}dz = 2\pi i[k_1+k_2] = 2\pi i\left[\frac{2+i}{6i}+\frac{1-2i}{12i}\right]$$

$$= \frac{2\pi i}{6i}\left[2+i+\frac{1-2i}{2}\right] = \frac{\pi}{6}[4+2i+1-2i] = \frac{5\pi}{6}$$

$$\therefore \int_{-\infty}^{\infty} \frac{x^2+x+3}{x^4+5x^2+4}dx = \frac{5\pi}{6}$$

EXAMPLE-25:

Evaluate:

$$\int_{0}^{\infty} \frac{\sin mx}{x}dx;\ \ m>0$$

SOLUTION:

Let

$$\int_{0}^{\infty} \frac{\sin mx}{x}dx = \int_{0}^{\infty} \frac{\sin mz}{z}dz$$

Now, we take

$$\int_{0}^{\infty} \frac{e^{imz}}{z}dz$$

For pole z=0 $\Rightarrow$ z_0=0

z_0=0 is first order pole.

$$k_1 = \lim_{z\to 0}(z-0)\cdot\frac{e^{imz}}{z} = e^{im(0)} = 1$$

By Cauchy residue theorem,

$$\int_{0}^{\infty} \frac{e^{imz}}{z}dz = \pi i(k_1)$$

$$\therefore \int_0^\infty \frac{\cos mz + i \sin mz}{z} dz = \pi i$$

$$\therefore \int_0^\infty \frac{\cos mz}{z} dz + i \int_0^\infty \frac{\sin mz}{z} dz = 0 + \pi i$$

Compare the real & imaginary part, we get

$$\int_0^\infty \frac{\cos mz}{z} dz = 0 \quad \& \quad \int_0^\infty \frac{\sin mz}{z} = \pi$$

EXERCISE-D

1) $\int_0^\infty \frac{dx}{x^2+1} = \frac{\pi}{2}$
2) $\int_{-\infty}^\infty \frac{dx}{(1+x^2)^2} = \frac{\pi}{2}$
3) $\int_0^\infty \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}$
4) $\int_0^\infty \frac{x^2}{(x^2+9)(x^2+4)^2} = \frac{\pi}{200}$
5) $\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}; a > 0$