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SUBJECT: MATHEMATICS

F.Y.B.Sc. SEM-1 (CBCS)

CH:2 INDETERMINANT FORM

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SHREE H.N. SHUKLA GROUP OF COLLEGES

CHAPTER 1 : "INDETERMINATE FORMS"

Definition: Suppose we want to find the limiting value of a function $F(x) = \frac{f(x)}{g(x)}$ at $x = a$, when direct substitution of $x = a$ gives the form $\frac{0}{0}$, i.e. at $x = a$, $f(x) = 0$ and $g(x) = 0$.

The form $\frac{0}{0}$ is called an indeterminate form. The other types of indeterminate forms are $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, 0^0, ∞^0 and 1^∞.

THE INDETERMINATE FORM $\frac{0}{0}$

L' HOSPITAL'S RULE:

Let the functions f and g be differentiable at every point than possibly $'\boldsymbol{a}'$ in some interval, with $g'(x) \neq 0$, if $x \neq a$.

If $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

Example – Evaluate $\boldsymbol{\lim_{x\to 0} \frac{e^x+e^{-x}-x^2-2}{sin^2x-x^2}}$

[3 Marks] [Step 1: 1 marks; Step 2: 1 marks; Step 3: 1 marks]

- **Solution:**
  - **Step 1:**

When $x \to 0$, both numerator and denominator approach to 0. Therefore we apply L' Hospital's rule.

$$\text{Let } L = \lim_{x\to 0} \frac{e^x + e^{-x} - x^2 - 2}{sin^2 x - x^2}$$

$$= \lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{sin\,2x - 2x} \left(\frac{0}{0}\right)$$

Step 2:

$$L = \lim_{x\to 0} \frac{e^x + e^{-x} - 2}{2cos2x - 2}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{e^x - e^{-x}}{-4sin2x}\left(\frac{0}{0}\right)$$

Step 3:

$$L = \lim_{x\to 0} \frac{e^x + e^{-x}}{-8cos2x}$$

$$= -\frac{1}{4}$$

Example – Evaluate $\lim_{x\to 1} \frac{x^x - x}{x-1-logx}$

⇨ **Solution:**

Example – Evaluate $\lim_{x\to 0} \frac{log(1-x^2)}{logcosx}$

⇨ **Solution:**

Example – Evaluate $\lim_{x\to 0} \frac{tanx - sinx}{sin^3x}$

⇨ **Solution:**

THE INDETERMINATE FORM $\frac{\infty}{\infty}$:

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} \frac{f(x)}{g(x)}$ becomes $\frac{\infty}{\infty}$

In this case we write $\frac{f(x)}{g(x)}$ as $\frac{1/g(x)}{1/f(x)}$ in which case the form reduces to $\frac{0}{0}$ and L' Hospital's rule is applicable.

Example: Evaluate $\lim_{x\to \frac{\pi}{2}} \frac{log(x-\frac{\pi}{2})}{tanx}$

[3 Marks] [Step 1: 1 mark; Step 2: 1 marks; Step 3: 1 marks]

⇨ **Solution:**

Step 1:

As x approaches to $\frac{\pi}{2}$, thefraction $\frac{log\,(x-\frac{\pi}{2})}{tanx}$assumes the indeterminate form $\frac{\infty}{\infty}$

.

$$\text{Let } L = \lim_{x\to\frac{\pi}{2}} \frac{log(x-\frac{\pi}{2})}{tanx}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\frac{1}{x-\frac{\pi}{2}}}{sec^2x}\left(\frac{0}{0}\right)$$

Step 2:

$$L = \lim_{x\to\frac{\pi}{2}} \frac{\frac{1}{x-\frac{\pi}{2}}}{\frac{1}{cos^2x}}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{cos^2x}{x-\frac{\pi}{2}}\left(\frac{0}{0}\right)$$

Step 3:

$$L = \lim_{x\to\frac{\pi}{2}} -2cosx.\,sinx$$

$$\therefore \mathrm{L} = 0$$

Example: **Evaluate**$\boldsymbol{\lim_{x\to\infty} x^n e^{-ax}}$**, n being a positive integer and** $\boldsymbol{a} > 0$**.**

⇨ **Solution:**

Example: **Evaluate**$\boldsymbol{\lim_{x\to a} \frac{log(x-a)}{log(e^x-e^a)}}$

⇨ **Solution:**

Example: **Evaluate**$\boldsymbol{\lim_{x\to 0} \log_{tanx} tan2x}$

⇨ **Solution:**

THE INDETERMINATE FORM $\mathbf{0} \times \infty$:

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} f(x).g(x)$ becomes $0 \times \infty$

In this case we write $f(x).g(x)$as $\frac{f(x)}{1/g(x)}$and the form reduces to $\frac{0}{0}$, Where L' Hospital's rule is applicable.

Example: Evaluate$\lim_{x\to\frac{\pi}{4}}(1 - \tan x)\sec 2x$

[3 Marks] [Step 1: 1 mark; Step 2: 2 marks]

⇨ **Solution:**

Step 1:

Let $L = \lim_{x\to\frac{\pi}{4}}(1 - \tan x)\sec 2x$

$$= \lim_{x\to\frac{\pi}{4}} \frac{1 - \tan x}{\cos 2x}\left(\frac{0}{0}\right)$$

Step 2:

$$L = \lim_{x\to\frac{\pi}{4}} \frac{-\sec^2 x}{-2\sin 2x}$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{-\sec^2 x}{-2\sin x.\cos x}$$

$\therefore L = 1$

Example: Evaluate$\lim_{x\to a} \log\left(2 - \frac{x}{a}\right)\cot(x - a)$

⇨ **Solution:**

Example: Evaluate$\lim_{x\to 1}(x^2 - 1)\tan\left(\frac{\pi x}{2}\right)$

⇨ **Solution:**

Example: Evaluate $\lim_{x\to 1} \log(1 - x)\cot\frac{\pi x}{2}$

⇨ **Solution:**

THE INDETERMINATE FORM $\infty - \infty$:

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} [f(x) - g(x)]$ becomes $\infty - \infty$

The form $\infty - \infty$ reduces to $\frac{0}{0}$ or $\frac{\infty}{\infty}$, for $f(x) - g(x) = \dfrac{\frac{1}{g(x)} - \frac{1}{f(x)}}{\frac{1}{f(x).g(x)}}$

When limits are taken, then the R.H.S. of above form is reduced to the form $\frac{0}{0}$ and L' Hospital's rule is applicable.

Example: Evaluate $\boldsymbol{\lim_{x\to 0} \frac{a}{x} - cot\frac{x}{a}}$

[4 Marks] [Step 1: 1 mark; Step 2: 1 marks; Step 2: 2 marks]

⇨ **Solution:**

Step 1:

Let $L = \lim_{x\to 0} \frac{a}{x} - cot\frac{x}{a}$

$= \lim_{t\to 0} \frac{1}{t} - \cot\ t$ $\quad (\infty - \infty)$, Where $t = \frac{x}{a}$

$= \lim_{t\to 0} \frac{\tan t - t}{t.\tan t} \left(\frac{0}{0}\right)$

Step 2:

$L = \lim_{t\to 0} \left(\frac{\tan t - t}{t^2}\right)\left(\frac{t}{\tan t}\right)$

$= \left(\lim_{t\to 0} \frac{\tan t - t}{t^2}\right)\left(\lim_{t\to 0} \frac{t}{\tan t}\right)$

$= \lim_{t\to 0} \frac{\tan t - t}{t^2} \left(\frac{0}{0}\right) \left[\because \lim_{t\to 0} \frac{t}{\tan t} = 0\right]$

Step 2:

$L = \lim_{t\to 0} \frac{sec^2 t - 1}{2t} \left(\frac{0}{0}\right)$

$= \lim_{t\to 0} \frac{tan^2 t}{2t} \left(\frac{0}{0}\right)$

$$= \lim_{t \to 0} \frac{1}{2} . \left(\frac{\tan t}{t}\right) \tan t$$

$$= \frac{1}{2}(1)(0) \left[\because \lim_{t \to 0} \frac{\tan t}{t} = 0\right]$$

$$= 0$$

<u>Example:</u> Evaluate $\mathbf{lim}_{x\to 1}\left(\frac{x}{x-1} - \frac{1}{logx}\right)$

⇨ **Solution:**

<u>Example:</u> Evaluate $\mathbf{lim}_{x\to 0}\left(\frac{1}{2x} - \frac{1}{x(e^{\pi x}+1)}\right)$

⇨ **Solution:**

THE INDETERMINATE FORMS 0^0, ∞^0, 1^∞:

Given $\lim_{x \to a}\{f(x)\}^{g(x)}$, where

$\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$,

Or

$\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = 0$

Or

$\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = \infty$

In this case put $y = \{f(x)\}^{g(x)}$

Taking logarithm on both sides, we get

$\log y = g(x) . \log f(x)$

Let $\lim_{x \to a} g(x) . \log f(x) = l$

Then $\lim_{x \to a} \log y = l$

Hence $\lim_{x \to a} \mathrm{y} = e^l$

$\lim_{x \to a}\{f(x)\}^{g(x)} = e^l$

Example: Evaluate $\lim_{x\to 0^+} x^{\sin x}$ (0^0)

[4 Marks] [Step 1: 1 mark; Step 2: 2 marks; Step 3: 1 marks]

⇨ **Solution:**

Step 1:

Let $y = x^{\sin x}$

$\therefore \log y = sinx . logx = \frac{\log x}{cosec\, x}$ has an indeterminate form $\frac{\infty}{\infty}$

$$\therefore \lim_{x\to 0^+} \log y = \lim_{x\to 0^+} \frac{\log x}{cosec\, x}$$

Step 2:

$$\therefore \lim_{x\to 0^+} \log y = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-cosec\, x\, .cotx}$$

$$= \lim_{x\to 0^+} \frac{sin^2 x}{-x\, .cosx}$$

$$= \lim_{x\to 0^+} \frac{2\sin x.\ \cos x}{-x.\ sin\ x - cosx}$$

$$= 0$$

Step 3:

$$\therefore \lim_{x\to 0^+} y = \lim_{x\to 0^+} x^{sin\, x} = e^0 = 1$$

$$\therefore \lim_{x\to 0^+} x^{sin\, x} = 1$$

Example: Evaluate $\lim_{x\to 1}(1 - x^2)^{\frac{1}{log(1-x)}}$ (0^0)

⇨ **Solution:**

Example: Evaluate $\lim_{x\to 0} \left(\frac{1}{x}\right)^{1-cos\, x}$ (∞^0)

[3 Marks] [Step 1: 1 mark; Step 2: 2 marks]

⇨ **Solution:**

Step 1:

Let $y = \left(\frac{1}{x}\right)^{1-cos\, x}$

$\therefore \log y = (1-\cos x)\ .\ \log\left(\frac{1}{x}\right)$

$$\lim_{x\to 0} log\ y = \lim_{x\to 0} -\left(2sin^2\ \frac{x}{2}\right)\ log\ x$$

$$= \lim_{x\to 0} -\frac{2\ log\ x}{cosec^2 \frac{x}{2}}$$

<u>Step 2:</u>

$$\lim_{x\to 0} \log \mathrm{y} = \lim_{x\to 0} \frac{-\frac{2}{x}}{-2\ cosec^2 \frac{x}{2}\ .\ cot \frac{x}{2}\ .\ \frac{1}{2}}$$

$$= \lim_{x\to 0} \frac{2\ sin^3 \frac{x}{2}}{x\ .\ cos \frac{x}{2}}$$

$$= \lim_{x\to 0} \left(\frac{\tan \frac{x}{2}}{\frac{x}{2}}\right) .sin^2 \frac{x}{2}$$

$$= 1 \times 0 \qquad \left[\because\ \lim_{x\to 0} \left(\frac{\tan\ \frac{x}{2}}{\frac{x}{2}}\right) = 1\right]$$

$$= 0$$

<u>**Example:**</u> **Evaluate** $\mathbf{lim}_{x\to 0}(\boldsymbol{cot\ x})^{\frac{1}{log\ x}}$ $\quad (\infty^{\mathbf{0}})$

⇨ **Solution:**

⇨ <u>**Example:**</u> **Evaluate** $\mathbf{lim}_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}\ (\mathbf{1})^{\infty}$

[3 Marks] [Step 1: 1 mark; Step 2: 2 marks]

⇨ **Solution:**

<u>Step 1:</u>

Let $L = \lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}$

$$\therefore log\ L = \lim_{x\to 0} \frac{\log(a^x + b^x + c^x) - \log 3}{x} \left(\frac{0}{0}\right)$$

Step 2:

$$\therefore\ log\ L = \lim_{x\to 0} \frac{a^x \log a \ + \ b^x \log b \ + \ c^x \log c}{a^x \ + \ b^x \ + \ c^x}$$

$$= \frac{1}{1+1+1} \ (\log a + \log b + \log c)$$

$$= \frac{1}{3} \log(abc)$$

$$\therefore \log L = \log(abc)^{\frac{1}{3}}$$

$$\therefore L = (abc)^{\frac{1}{3}}$$

Example: **Evaluate**$\boldsymbol{\lim_{x\to 0} \left(\frac{sinh\ x}{x}\right)^{\frac{1}{x^2}}\ (1)^{\infty}}$

⇨ **Solution:**

ASSIGNMENT

Q. 1 $\lim_{x\to 0}\frac{1}{x}(1-xcotx)$

Q. 2 $\lim_{\theta\to\alpha}\frac{1-cos(\theta-\alpha)}{(sin\theta-sin\alpha)^2}$

Q. 3 $\lim_{x\to a}\frac{log(x-a)}{log(a^x-a^a)}$

Q. 4 $\lim_{x\to y}\frac{x^y-y^x}{x^x-y^y}$

Q. 5 $\lim_{x\to\infty}(cosh^{-1}x-logx)$

Q. 6 $\lim_{x\to 0}\frac{a^x-b^x}{x}$

Q. 7 $\lim_{x\to\infty}x^2e^{-x}$

Q. 8 $\lim_{x\to 0}(1+sinx)^{cotx}$

Q. 9 $\lim_{x\to\infty}\left(\frac{ax+1}{ax-1}\right)^x$

Q.10 $\lim_{x\to 0}(\sin x)^{\tan x}$

Q. 11 $\lim_{x\to 0}(a^x+x)^{\frac{1}{x}}$

Q.12 $\lim_{x\to 0}(\cos x)^{\frac{1}{x^2}}$

Q. 13 $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$