SHREE H.N.SHUKLA COLLEGE OF SCIENCE

S.Y.B.Sc. SEM-4
PAPER 401
LINEAR ALGEBRA, REAL ANALYSIS
& DIFFERENTIAL GEOMETRY



Linear Transformations

Linear Transformations Transformations of Euclidean

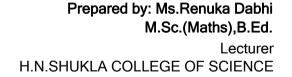
Kernel and Range

a linear trans.

Composition of linear trans.

Kernel and Range

Linear Transformation





| Linear Trans- formations | | Agenda |
|--|--|--------|
| Linear Trans- formations Transformations of Euclidean space Kernel and Range The matrix of a linear trans. Composition of linear trans. Kernel and Range | Linear Transformations Linear transformations of Euclidean space Kernel and Range | |
| | The matrix of a linear transformation Composition of linear transformations | |

Kernel and Range



formations

Linear Transformations of Euclidean space Kernel and Range The matrix of a linear trans.

In the $m \times n$ linear system we can regard A astransforming elements of \mathbb{R}^n (as column

vectors) into elements of
$$\mathbb{R}^m$$
 via the rule

Then solving the system amounts to finding all of the vectors $\mathbf{x} \in \mathbb{R}^n$ such that $T(\mathbf{x}) = \mathbf{0}$.

 $A\mathbf{x} = \mathbf{0}$.

 $T(\mathbf{x}) = A\mathbf{x}$

Motivation

Solving the differential equation
$$y^{JJ} + y = 0$$
 is equivalent to finding functions y su

is equivalent to finding functions y such that T(y) = 0, where T is defined as $T(y) = y^{JJ} + y$.

Range 1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and The matrix of a linear trans. 2. $T(c\mathbf{v}) = cT(\mathbf{v})$ for all vectors $\mathbf{u}, \mathbf{v} \in V$ and all scalars c. V is called the Range

Linear Trans-

space

Kernel and

Examples

 $m \times n$ matrix

domain and W the **codomain** of T.

 $ightharpoonup T: \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{x}) = A\mathbf{x}$, where A is an

 $ightharpoonup T: C^{k}(I) \rightarrow C^{k-2}(I)$ defined by $T(y) = y^{JJ} + y$

 $ightharpoonup T: M_{m \times n}(R) \to M_{n \times m}(R)$ defined by $T(A) = A^T$

► T: $P_1 \to P_2$ defined by $T(a + bx) = (a + 2b) + 3ax + 4bx^2$

formations

Examples

Linear Transformations Transformation

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of Euclidean space

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linear trans.

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- 1. Verify that $T: M_{m \times n}(\mathbb{R}) \to M_{n \times m}(\mathbb{R})$, where $T(A) = A^T$, is a linear transformation.
 - The transpose of an $m \times n$ matrix is an $n \times m$ matrix.
 - If $A, B \in M_{m \times n}(\mathbb{R})$, then

$$T(A + B) = (A + B)^T = A^T + B^T = T(A) + T(B).$$

If $A \in M_{m \times n}$ (R) and $c \in R$, then

2. Verify that $T: C^k(I) \to C^{k-2}(I)$, where $T(y) = y^{jj} + y$,

$$T(cA) = (cA)^{\mathsf{T}} = cA^{\mathsf{T}} = cT(A).$$

- is a linear transformation.

 If $y \in C^k(I)$ then $T(y) = y'' + y \in I$
 - C^{k-} $A(1)_1, y_2 \in C^k(1)$, then

$$T(y_1 + y_2) = (y_1 + y_2)'' + (y_1 + y_2) = y_1'' + y_2'' + y_1 + y_2$$

$$= (y_1'' + y_1) + (y_2'' + y_2) = T(y_1) + T(y_2).$$

If
$$y \in C^k(I)$$
 and $c \in R$, then
$$T(cy) = (cy)'' + (cy) = cy'' + cy = c(y'' + y) = cT(y).$$



Range

Range



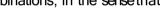


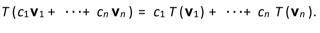
In particular, if
$$\{\mathbf{v}_1, ..., \mathbf{v}_n\}$$

then knowing
$$T(\mathbf{v}_1), \dots$$

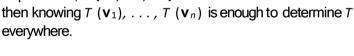
In particular, if
$$\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$$
 then knowing $T(\mathbf{v}_1), \dots, T$

Specifying linear transformations





In particular, if $\{v_1, \ldots, v_n\}$ is a basis for the domain of T,





Range

The matrix of a linear trans.

Kernel and

Range

Linear Trans-

formations

Linear transformations from \mathbb{R}^n to \mathbb{R}^m

Let A be an $m \times n$ matrix with real entries and define $T: \mathbb{R}^n \to \mathbb{R}^m$ by $T(\mathbf{x}) = A\mathbf{x}$. Verify that T is a linear transformation.

- If **x** is an $n \times 1$ column vector then A **x** is an $m \times 1$ column vector.
- $T(\mathbf{x} + \mathbf{y}) = A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = T(\mathbf{x}) + T(\mathbf{y})$
- $T(c\mathbf{x}) = A(c\mathbf{x}) = cA\mathbf{x} = cT(\mathbf{x})$

Such a transformation is called a matrix transformation. In fact, every linear transformation from R^n to R^m is a matrix transformation.



Range

Linear Trans-

formations

Theorem Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is

Example

defined by

Determine the matrix of the linear transformation $T: \mathbb{R}^4 \to \mathbb{R}^3$

and $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ denote the standard basis vectors for \mathbb{R}^n . This A is called the matrix of T.

 $A = T(\mathbf{e}_1) T(\mathbf{e}_2) \cdots T(\mathbf{e}_n)$

described by the matrix transformation $T(\mathbf{x}) = A\mathbf{x}$, where

 $T(x_1, x_2, x_3, x_4) = (2x_1 + 3x_2 + x_4, 5x_1 + 9x_3 - x_4,$

 $4x_1 + 2x_2 - x_3 + 7x_4$).

Matrix transformations

Kernel and

a linear trans.

Range The matrix of

Range

consisting of all the vectors $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \mathbf{0}$ is called the **kernel** of T It is denoted $Ker(T) = \{ \mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0} \}.$

Kernel

Example

Let $T: C^k(I) \to C^{k-2}(I)$ be the linear transformation

 $T(y) = y^{jj} + y$. Its kernel is spanned by $\{\cos x, \sin x\}$.

Remarks

- The kernel of a linear transformation is a subspace of its domain.
- The kernel of a matrix transformation is simply the null space of the matrix.

Range

Range

The matrix of a linear trans.

The **range** of the linear transformation $T: V \rightarrow W$ is the

 $Rng(T) = \{T(\mathbf{v}) \in W : \mathbf{v} \in V\}.$

Range

Example

Consider the linear transformation $T: M_n(\mathbb{R}) \to M_n(\mathbb{R})$

defined by $T(A) = A + A^{T}$. The range of T is the subspace

of symmetric $n \times n$ matrices.

Remarks

The range of a linear transformation is a subspace of its

codomain.

The range of a matrix transformation is the column space

of the matrix

Range

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formations

Linear Trans-

We know

Theorem

ightharpoonup Rng(T) = colspace(A), ightharpoonup dim (Rng(T)) = rank(A), ▶ the domain of T is Rⁿ. \rightarrow dim (domain of T) = n. We know from the rank-nullity theorem that

Ker(T) = nullspace(A),

Suppose T is the matrix transformation with $m \times n$ matrix A.

rank(A) + nullitv(A) = n.

This fact is also true when T is not a matrix transformation:

Hence.

Rank-Nullity revisited

 \rightarrow dim (Ker(T)) = nullity(A),



If $T: V \to W$ is a linear transformation and V is finite-dimensional, then

 $\dim(\operatorname{Ker}(T)) + \dim(\operatorname{Rng}(T)) = \dim(V).$

Theorem

Let V be a vector space with basis $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$. Then every vector $\mathbf{v} \in V$ can be written in a unique way as a linear combination

$$\mathbf{V} = c_1 \mathbf{V}_1 + c_2 \mathbf{V}_2 + \cdots + c_n \mathbf{V}_n.$$

In other words, picking a basis for a vector space allows us to give coordinates for points. This will allow us to give matrices for linear transformations of vector spaces besides \mathbb{R}^n .



The matrix of

a linear trans.

Range

Linear Transformations

Definition

Let V and W be vector spaces with ordered bases $B = \{ \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \} \text{ and } C = \{ \mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m \},$ respectively, and let $T: V \to W$ be a linear transformation.

The matrix of a linear transformation

The matrix representation of T relative to the bases Band C is

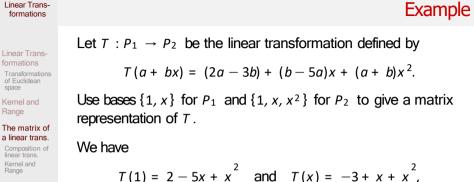
$$A = [a_{ij}]$$

where

$$T(\mathbf{v}_i) = a_{1i} \mathbf{w}_1 + a_{2i} \mathbf{w}_2 + \cdots + a_{mi} \mathbf{w}_m.$$

In other words, A is the matrix whose j-th column is $T(\mathbf{v}_i)$, expressed in coordinates using $\{\mathbf{w}_1, \dots, \mathbf{w}_m\}$.



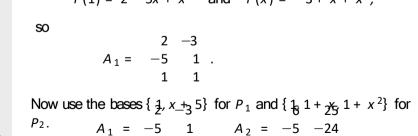


space

Range

Range

We have



Linear Transformations Transformation of Euclidean space Kernel and Range

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Composition of

Linear Trans-

formations

Definition Let $T_1: U \to V$ and $T_2: V \to W$ be linear transformations.

Their **composition** is the linear transformation $T_2 \circ T_1$ defined by $(T_2 \circ T_1) (\mathbf{u}) = T_2 (T_1(\mathbf{u}))$.

Composition of linear transformations

Theorem

Let T_1 and T_2 be as above, and let B, C, and D be ordered bases for U, V, and W, respectively. If

- ► A₁ is the matrix representation for T₁ relative to B and C,
- $ightharpoonup A_2$ is the matrix representation for T_2 relative to C and D,
- A_{21} is the matrix representation for T_2 relative to C and D,

and D, then $A_{21} = A_2 A_1$.



The inverse of a linear transformation

exists) is a linear transformation $T^{-1}: W \to V$ such that

Let T be as above and let A be the matrix representation of T relative to bases B and C for V and W, respectively. T has an inverse transformation if and only if A is invertible and, if so, T^{-1} is the linear transformation with matrix A^{-1} relative to C

 $T^{-1} \circ T$ (v) = v and $T \circ T^{-1}$ (w) = w

Definition

Theorem

and B.

for all $\mathbf{v} \in V$ and $\mathbf{w} \in W$.

If $T:V\to W$ is a linear transformation, its **inverse** (if it

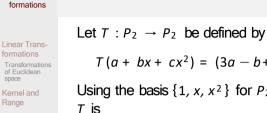
of Euclidean Kernel and

The matrix of a linear trans.

Composition of linear trans Kernel and Range

Range

Linear Trans-



$T(a + bx + cx^2) = (3a - b + c) + (a - c)x + (4b + c)x^2$ Using the basis $\{1, x, x^2\}$ for P_2 , the matrix representation for

The matrix of a linear trans. Composition of linear trans. Range

This matrix is invertible and
$$A^{-1} = \frac{1}{17}$$

$$= \frac{-}{17} \quad \frac{-1}{4} \quad \frac{3}{-12} \quad \frac{4}{1}$$

Example

Thus,
$$T^{-1}$$
 is given by
$$T^{-1}(a + bx + cx^{2}) = \frac{4a+5b+c}{17} + \frac{-a+3b+4c}{17}x + \frac{4a-12b+c}{17}x^{2}.$$

Theorem

Let $T: V \to W$ be a linear transformation and A be a matrix representation of T relative to some bases for V and W.

- $\mathsf{Ker}(T) = \{c_1 \mathbf{v}_1 + \cdots + c_n \mathbf{v}_n \in V : (c_1, \dots, c_n) \in \mathsf{nullspace}(A)\}.$
- $Rng(T) = \{c_1\mathbf{W}_1 + \cdots + c_m\mathbf{W}_m \in W : (c_1, \ldots, c_m) \in colspace(A)\}.$



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